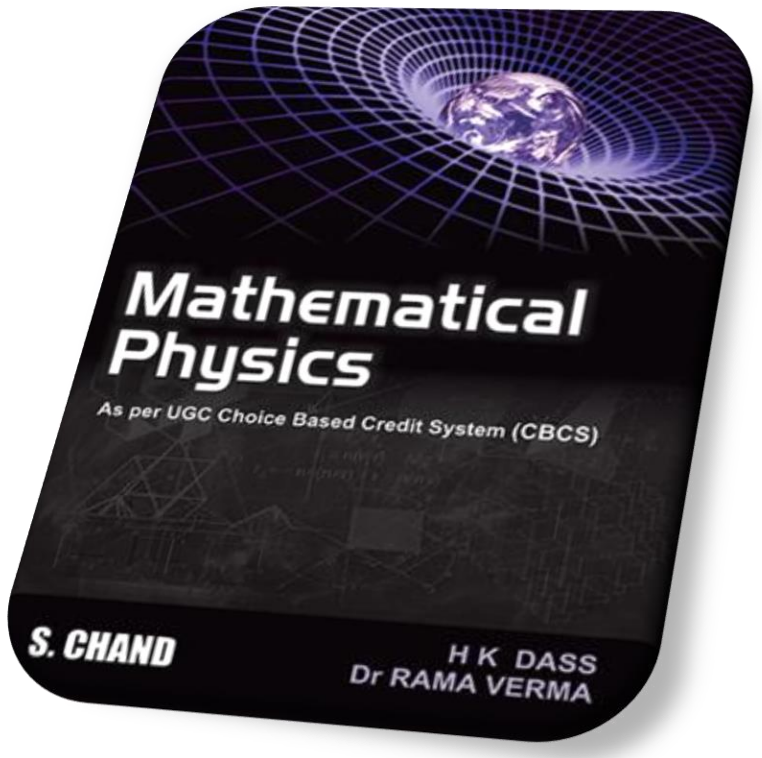

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31 CHAPTER

Unit-XII: Matrices

Matrices

31.1 DEFINITION

Let us consider a set of simultaneous equations,

$$\begin{aligned} x + 2y + 3z + 5t &= 0 \\ 4x + 2y + 5z + 7t &= 0 \\ 3x + 4y + 2z + 6t &= 0. \end{aligned}$$

Now we write down the coefficients of x, y, z, t of the above equations and enclose them within brackets and then we get

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 4 & 2 & 5 & 7 \\ 3 & 4 & 2 & 6 \end{bmatrix}$$

The above system of numbers, arranged in a rectangular array in rows and columns and bounded by the brackets, is called a matrix.

It has got 3 rows and 4 columns and in all $3 \times 4 = 12$ elements. It is termed as 3×4 matrix, to be read as [3 by 4 matrix]. In the double subscripts of an element, the first subscript determines the row and the second subscript determines the column in which the element lies, a_{ij} lies in the ith row and jth column.

31.2 VARIOUS TYPES OF MATRICES

(*i*) **Row Matrix.** If a matrix has only one row and any number of columns, it is called a *Row matrix*, *e.g.,* [2 7 3 9]

(*b*) **Column Matrix.** A matrix, having one column and any number of rows, is called a *Column matrix*, *e.g.,* $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(*c*) **Null Matrix or Zero Matrix.** Any matrix, in which all the elements are zeros, is called a *Zero matrix* or *Null matrix e.g.,*

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(*d*) **Square Matrix.** A matrix, in which the number of rows is equal to the number of columns, is called a square matrix *e.g.,*

$$\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$$

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(*e*) **Diagonal Matrix.** A square matrix is called a diagonal matrix, if all its non-diagonal elements are zero *e.g.,*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(*f*) **Scalar matrix.** A diagonal matrix in which all the diagonal elements are equal to a scalar, say (*k*) is called a scalar matrix.

For example;

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} -6 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

i.e., $A = [a_{ij}]_{n \times n}$ is a scalar matrix if $a_{ij} = \begin{cases} 0, & \text{when } i \neq j \\ k, & \text{when } i = j \end{cases}$

(*g*) **Unit or Identity Matrix.** A square matrix is called a unit matrix if all the diagonal elements are unity and non-diagonal elements are zero *e.g.,*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(*h*) **Symmetric Matrix.** A square matrix will be called symmetric, if for all values of *i* and *j*, $a_{ij} = a_{ji}$ *i.e.,* $A' = A$

$$e.g., \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

(*i*) **Skew Symmetric Matrix.** A square matrix is called skew symmetric matrix, if

(1) $a_{ij} = -a_{ji}$ for all values of *i* and *j*, or $A' = -A$

(2) All diagonal elements are zero, *e.g.,*

$$\begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}$$

(*j*) **Triangular Matrix.** (Echelon form) A square matrix, all of whose elements below the leading diagonal are zero, is called an *upper triangular matrix.* A square matrix, all of whose elements above the leading diagonal are zero, is called *a lower triangular matrix e.g.,*

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 6 & 7 \end{bmatrix}$$

Upper triangular matrix Lower triangular matrix

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(*k*) **Transpose of a Matrix.** If in a given matrix A, we interchange the rows and the corresponding columns, the new matrix obtained is called the transpose of the matrix A and is denoted by A' or A^T *e.g.*,

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 5 \\ 6 & 7 & 8 \end{bmatrix}, \quad A' = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 0 & 7 \\ 4 & 5 & 8 \end{bmatrix}$$

(*l*) **Orthogonal Matrix.** A square matrix A is called an orthogonal matrix if the product of the matrix A and the transpose matrix A' is an identity matrix *e.g.*,

$$A.\, A' = I$$

if $|A| = 1$, matrix A is proper.

(*m*) **Conjugate of a Matrix**

Let
$$A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$$

Conjugate of matrix A is $\overline{A}$

$$\overline{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & i & 3+2i \end{bmatrix}$$

(*n*) **Matrix A$^\theta$.** Transpose of the conjugate of a matrix A is denoted by A^θ.

Let
$$A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$$

$$\overline{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & +i & 3+2i \end{bmatrix}$$

$$(\overline{A})' = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3+2i \end{bmatrix}$$

$$A^\theta = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3+2i \end{bmatrix}$$

(*o*) **Unitary Matrix.** A square matrix A is said to be unitary if

$$A^\theta A = I$$ **(*Vidyasagar University 2018*)**

e.g.
$$A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}, \quad A^\theta = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}, \quad A \cdot A^\theta = I$$

(*p*) **Hermitian Matrix.** A square matrix $A = (a_{ij})$ is called Hermitian matrix, if every *i-jth* element of A is equal to conjugate complex *j-i*th element of A.

In other words, $a_{ij} = \overline{a}_{ji}$

e.g.
$$\begin{bmatrix} 1 & 2+3i & 3+i \\ 2-3i & 2 & 1-2i \\ 3-i & 1+2i & 5 \end{bmatrix}$$

Necessary and sufficient condition for a matrix A to be Hermitian is that $A = A^\theta$ *i.e.* conjugate transpose of A

$$\Rightarrow \qquad A = (\overline{A})'.$$

(*q*) **Skew Hermitian Matrix.** A square matrix $A = (a_{ij})$ will be called a Skew Hermitian matrix if every *i-j*th element of A is equal to negative conjugate complex of *j-i*th element of A.

In other words, $$a_{ij} = -\overline{a}_{ji}$$

All the elements in the principal diagonal will be of the form

$$a_{ii} = -\overline{a}_{ii} \quad \text{or} \quad a_{ii} + \overline{a}_{ii} = 0$$

If $$a_{ii} = a + ib \quad \text{then} \quad \overline{a}_{ii} = a - ib$$

$$(a + ib) + (a - ib) = 0 \quad \Rightarrow \quad 2\,a = 0 \Rightarrow a = 0$$

So, a_{ii} is pure imaginary $\Rightarrow$ $a_{ii} = 0$.

Hence, all the diagonal elements of a Skew Hermitian Matrix are either zeros or pure imaginary.

e.g. $$\begin{bmatrix} i & 2-3i & 4+5i \\ -(2+3i) & 0 & 2i \\ -(4-5i) & 2i & -3i \end{bmatrix}$$

The necessary and sufficient condition for a matrix A to be Skew Hermitian is that

$$A^\theta = -A, \quad (\overline{A})' = -A$$

(*r*) **Idempotent Matrix.** A matrix, such that $A^2 = A$ is called Idempotent Matrix.

e.g. $$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}, A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

(*s*) **Periodic Matrix.** A matrix A will be called a Periodic Matrix, if

$$A^{k+1} = A$$

where k is a +ve integer. If k is the least + ve integer, for which $A^{k+1} = A$, then k is said to be the period of A. If we choose $k = 1$, we get $A^2 = A$ and we call it to be idempotent matrix.

(*t*) **Nilpotent Matrix.** A matrix will be called a Nilpotent matrix, if $A^k = 0$ (null matrix) where k is a +ve integer ; if however k is the least +ve integer for which $A^k = 0$, then k is the *index* of the nilpotent matrix.

e.g., $$A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}, A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

A is nilpotent matrix whose index is 2.

(*u*) **Involuntary Matrix.** A matrix A will be called an Involuntary matrix, if $A^2 = I$ (unit matrix). Since $I^2 = I$ always $\therefore$ Unit matrix is involuntary.

(*v*) **Equal Matrices.** Two matrices are said to be equal if

(*i*) They are of the same order.

(*ii*) The elements in the corresponding positions are equal.

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Thus if $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Here $A = B$

(w) **Singular Matrix.** If the determinant of the matrix is zero, then the matrix is known as singular matrix *e.g.* $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is singular matrix, because $|A| = 6 - 6 = 0$.

Example. *Find the values of x, y, z and 'a' which satisfy the matrix equation.*

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$$

Solution. As the given matrices are equal, so their corresponding elements are equal.

$$x + 3 = 0 \quad \Rightarrow \quad x = -3 \qquad \ldots (1)$$
$$2y + x = -7 \qquad \ldots (2)$$
$$z - 1 = 3 \quad \Rightarrow \quad z = 4 \qquad \ldots (3)$$
$$4a - 6 = 2a \quad \Rightarrow \quad a = 3 \qquad \ldots (4)$$

Putting the value of $x = -3$ from (1) into (2), we have

$$2y - 3 = -7 \quad \Rightarrow \quad y = -2$$

Hence, $x = -3, \quad y = -2, \quad z = 4, \quad a = 3$ **Ans.**

31.3 ADDITION OF MATRICES

If A and B be two matrices of the same order, then their sum, $A + B$ is defined as the matrix, each element of which is the sum of the corresponding elements of A and B.

Thus if $A = \begin{bmatrix} 4 & 2 & 5 \\ 1 & 3 & -6 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}$

then $A + B = \begin{bmatrix} 4+1 & 2+0 & 5+2 \\ 1+3 & 3+1 & -6+4 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 7 \\ 4 & 4 & -2 \end{bmatrix}$

If $A = [a_{ij}], \; B = [b_{ij}]$ then $A + B = [a_{ij} + b_{ij}]$

Symmetric and Anti Symmetric matrices

$$\boxed{A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')}$$

Square matrix = Symmetric matrix + Anti-symmetric matrix **Proved.**

Example 1. *Write matrix A given below as the sum of a symmetric and a skew symmetric matrix.*

$$A = \begin{pmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{pmatrix}$$

Solution. $A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix}$ On transposing, we get $A' = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 5 & 6 \\ 4 & 3 & 3 \end{bmatrix}$

On adding A and A′, we have

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$$A + A' = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 & -1 \\ 2 & 5 & 6 \\ 4 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 10 & 9 \\ 3 & 9 & 6 \end{bmatrix} \quad \ldots (1)$$

On subtracting A' from A, we get

$$A - A' = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -2 & -1 \\ 2 & 5 & 6 \\ 4 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 5 \\ -4 & 0 & -3 \\ -5 & 3 & 0 \end{bmatrix} \quad \ldots (2)$$

On adding (1) and (2), we have

$$2A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 10 & 9 \\ 3 & 9 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 5 \\ -4 & 0 & -3 \\ -5 & 3 & 0 \end{bmatrix}$$

$$\text{A} = \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 5 & \frac{9}{2} \\ \frac{3}{2} & \frac{9}{2} & 3 \end{bmatrix} + \begin{bmatrix} 0 & 2 & \frac{5}{2} \\ -2 & 0 & -\frac{3}{2} \\ \frac{-5}{2} & \frac{3}{2} & 0 \end{bmatrix}$$

A = [Symmetric matrix] + [Skew symmetric matrix.] **Ans.**

Example 2. *Express* $A = \begin{bmatrix} 1 & -2 & -3 \\ 3 & 0 & 5 \\ 5 & 6 & 1 \end{bmatrix}$ *as the sum of a lower triangular matrix and upper triangular matrix.*

Solution. Let $A = L + U$

$$\begin{bmatrix} 1 & -2 & -3 \\ 3 & 0 & 5 \\ 5 & 6 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} + \begin{bmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -3 \\ 3 & 0 & 5 \\ 5 & 6 & 1 \end{bmatrix} = \begin{bmatrix} a+1 & 0+p & 0+q \\ b+0 & c+1 & 0+r \\ d+0 & e+0 & f+1 \end{bmatrix}$$

Equating the corresponding elements on both the sides, we get

$$a + 1 = 1 \qquad p = -2 \qquad q = -3$$
$$b = 3 \qquad c + 1 = 0 \qquad r = 5$$
$$d = 5 \qquad e = 6 \qquad f + 1 = 1$$

On solving these equations, we get

$$a = 0 \qquad p = -2 \qquad q = -3$$
$$b = 3 \qquad c = -1 \qquad r = 5$$

$d = 5$ $e = 6$ $f = 0$

Hence, $L = \begin{bmatrix} 0 & 0 & 0 \\ 3 & -1 & 0 \\ 5 & 6 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ **Ans.**

31.4 PROPERTIES OF MATRIX ADDITION

Only matrices of the same order can be added or subtracted.

(*i*) **Commutative Law.** $A + B = B + A$. (*ii*) **Associative law.** $A + (B + C) = (A + B) + C$.

31.5 SUBTRACTION OF MATRICES

The difference of two matrices is a matrix, each element of which is obtained by subtracting the elements of the second matrix from the corresponding element of the first.

$$A - B = [a_{ij} - b_{ij}]$$

Thus
$$\begin{bmatrix} 8 & 6 & 4 \\ 1 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 5 & 1 \\ 7 & 6 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8-3 & 6-5 & 4-1 \\ 1-7 & 2-6 & 0-2 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 3 \\ -6 & -4 & -2 \end{bmatrix}$$ **Ans.**

31.6 SCALAR MULTIPLE OF A MATRIX

If a matrix is multiplied by a scalar quantity *k*, then each element is multiplied by *k*, *i.e.*

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix}$$

$$3A = 3\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 6 & 9 & 12 \\ 12 & 15 & 18 \\ 18 & 21 & 27 \end{bmatrix}$$

EXERCISE 31.1

1. (*a*) If $A = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$, represent it as A = B + C where B is a symmetric and C is a skew-symmetric matrix.

(*b*) Express $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 7 & 1 \\ 5 & 9 & 3 \end{bmatrix}$ as a sum of symmetric and skew-symmetric matrix.

2. Matrices A and B are such that

$3A - 2B = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$ and $-4A + B = \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix}$, Find A and B.

3. Given $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$, Find x, y, z and w.

4. If $A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, Find (*i*) $2A + 3B$ (*ii*) $3A - 4B$.

ANSWERS

1. (*a*) $A = \begin{bmatrix} -1 & \frac{9}{2} & 3 \\ \frac{9}{2} & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & \frac{5}{2} & -2 \\ \frac{-5}{2} & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$ (*b*) $A = \begin{bmatrix} 1 & \frac{5}{2} & \frac{5}{2} \\ \frac{5}{2} & 7 & 5 \\ \frac{5}{2} & 5 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & 0 & -4 \\ \frac{5}{2} & 4 & 0 \end{bmatrix}$

2. $A = \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 \\ 4 & -1 \end{bmatrix}$

3. $x = 2,\ y = 4,\ z = 1,\ w = 3$

4. (*i*) $\begin{bmatrix} 3 & 10 & 3 \\ 8 & 3 & 6 \\ 2 & 2 & 13 \end{bmatrix}$, (*ii*) $\begin{bmatrix} -4 & -2 & -4 \\ -5 & -4 & 9 \\ 3 & 3 & -6 \end{bmatrix}$

31.7 MULTIPLICATION

The product of two matrices A and B is only possible if the number of columns in A is equal to the number of rows in B.

Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be an $n \times p$ matrix. Then the product AB of these matrices is an $m \times p$ matrix $C = [c_{ij}]$ where

$$c_{ij} = a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + a_{i3}\, b_{3j} + ... + a_{in}\, b_{nj}$$

(AB)′ = B′A′

If A and B are two matrices conformal for product AB, then $(AB)' = B'A'$, *where dash represents transpose of a matrix.*

31.8 PROPERTIES OF MATRIX MULTIPLICATION

1. Multiplication of matrices is not commutative.

$$AB \neq BA$$

2. Matrix multiplication is associative, if conformability is assured.

$A\,(BC) = (AB)\,C$

3. Matrix multiplication is distributive with respect to addition.

$A\,(B + C) = AB + AC$

4. Multiplication of matrix A by unit matrix.

$AI = IA = A$

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5. Multiplicative inverse of a matrix exists if $|A| \neq 0$.

$A \,.\, A^{-1} = A^{-1} \,.\, A = I$

6. If A is a square then $A \times A = A^2$, $A \times A \times A = A^3$.

7. $A^0 = I$

8. $I^n = I$, where n is positive integer.

Example 1. *If* $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

from the products AB and BA, and show that AB ≠ BA.

Solution. Here,

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-0+3 & 0-2+6 & 2-4+0 \\ 2+0-1 & 0+3-2 & 4+6-0 \\ -3+0+2 & 0+1+4 & -6+2+0 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 1 & 1 & 10 \\ -1 & 5 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1+0-6 & -2+0+2 & 3-0+4 \\ 0+2-6 & 0+3+2 & 0-1+4 \\ 1+4+0 & -2+6+0 & 3-2+0 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 7 \\ -4 & 5 & 3 \\ 5 & 4 & 1 \end{bmatrix}$$

$AB \neq BA$ **Proved.**

Example 2. *Verify that*

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \text{ is orthogonal.}$$

Solution.

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \quad \therefore \quad A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$AA' = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, A is an orthogonal matrix. **Verified.**

EXERCISE 31.2

1. Compute AB, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 6 & 4 \\ 4 & 7 & 5 \end{bmatrix}$$

2. If $A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$. From the product AB and BA. Show that $AB \neq BA$.

3. If $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

4. If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ choose α and β so that $(\alpha I + \beta A)^2 = A$

5. Write the following transformation in matrix form :

$$x_1 = \frac{\sqrt{3}}{2} y_1 + \frac{1}{2} y_2 \;;\; x_2 = -\frac{1}{2} y_1 + \frac{\sqrt{3}}{2} y_2$$

Hence, find the transformation in matrix form which expresses y_1, y_2 in terms of x_1, x_2.

6. If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is a unit matrix, show that $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

7. If $f(x) = x^3 - 20x + 8$, find $f(A)$ where $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

8. Show that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}^{-1}$

9. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ then show that $A^3 = A^{-1}$.

10. Verify whether the matrix $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$ is orthogonal.

11. If A and B are square matrices of the same order, explain in general

(*i*) $(A + B)^2 \neq A^2 + 2AB + B^2$ (*ii*) $(A - B)^2 \neq A^2 - 2AB + B^2$ (*iii*) $(A + B)(A - B) \neq A^2 - B^2$

ANSWERS

1. $\begin{bmatrix} 20 & 38 & 26 \\ 47 & 92 & 62 \end{bmatrix}$

4. $\alpha = \beta = \pm\frac{1}{\sqrt{2}}$

5. $y_1 = \frac{\sqrt{3}}{2} x_1 - \frac{1}{2} x_2$, $y_2 = \frac{1}{2} x_1 + \frac{\sqrt{3}}{2} x_2$

7. 0

31.9 ADJOINT OF A SQUARE MATRIX

Let the determinant of the square matrix A be $|A|$.

If $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, Than $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

The matrix formed by the co-factors of the elements in

$$|A| \quad \text{is} \quad \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

where $A_1 = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} = b_2c_3 - b_3c_2$, $\quad A_2 = -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} = -b_1c_3 + b_3c_1$

$A_3 = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = b_1c_2 - b_2c_1$, $\quad B_1 = -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} = -a_2c_3 + a_3c_2$

$B_2 = \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} = a_1c_3 - a_3c_1$, $\quad B_3 = -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} = -a_1c_2 + a_2c_1$

$C_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2b_3 - a_3b_2$, $\quad C_2 = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = -a_1b_3 + a_3b_1$

$$C_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Then the transpose of the matrix of co-factors

$$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$$

is called the adjoint of the matrix A and is written as adj A.

31.10 PROPERTY OF ADJOINT MATRIX

The product of a matrix A and its adjoint is equal to unit matrix multiplied by the determinant A.

INVERSE OF A MATRIX

If A and B are two square matrices of the same order, such that

$$AB = BA = I \qquad (I = \text{unit matrix})$$

then B is called the inverse of A *i.e.* $B = A^{-1}$ and A is the inverse of B.

Condition for a square matrix A to possess an inverse is that matrix A is non-singular, i.e., $|A| \neq 0$

If A is a square matrix and B be its inverse, then $AB = I$

Taking determinant of both sides, we get

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$$|AB| = |I| \text{ or } |A|\,|B| = I$$

From this relation it is clear that $|A| \neq 0$ *i.e.* the matrix A is non-singular.

To find the inverse matrix with the help of adjoint matrix

We know that $\quad A\,.\,(Adj.\,A) = |A|\,I$

$$\Rightarrow \quad A \cdot \frac{1}{|A|}(Adj.\,A) = I \qquad \text{[Provided } |A| \neq 0\text{]} \quad \ldots (1)$$

and $\quad A\,.\,A^{-1} = I \qquad \ldots (2)$

From (1) and (2), we have

$$\therefore \quad \boxed{A^{-1} = \frac{1}{|A|}(Adj.\,A)}$$

Example 1. *If* $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, *find* A^{-1}.

Solution. $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$$|A| = 3(-3+4) + 3(2-0) + 4(-2-0) = 3 + 6 - 8 = 1$$

The co-factors of elements of various rows of $|A|$ are

$$\begin{bmatrix} (-3+4) & (-2-0) & (-2-0) \\ (3-4) & (3-0) & (3-0) \\ (-12+12) & (-12+8) & (-9+6) \end{bmatrix}$$

Therefore, the matrix formed by the co-factors of $|A|$ is

$$\begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}, \quad Adj.A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} Adj.A = \frac{1}{1}\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$ **Ans.**

Example 2. *If A and B are non-singular matrices of the same order then,*

$$(AB)^{-1} = B^{-1}.A^{-1}$$

Hence prove that $(A^{-1})^m = (A^m)^{-1}$ *for any positive integer m.*

Solution. We know that,

$$(AB)\,.\,(B^{-1}A^{-1}) = [(AB)\,B^{-1}]\,.\,A^{-1} = [A\,(BB^{-1}]\,.\,A^{-1}$$
$$= [AI]\,A^{-1} = A\,.\,A^{-1} = I$$

Also, $\quad B^{-1}A^{-1}\,.\,(AB) = B^{-1}[A^{-1}\,.\,(AB)] = B^{-1}\,[(A^{-1}A)\,.\,B]$

$$= B^{-1}\,[I\,.\,B] = B^{-1}\,.\,B = I$$

By definition of the inverse of a matrix, $B^{-1}A^{-1}$ is inverse of AB.

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$$\Rightarrow \qquad B^{-1} A^{-1} = (AB)^{-1}$$ **Proved.**

$$(A^m)^{-1} = [A \,.\, A^{m-1}]^{-1} = (A^{m-1})^{-1} A^{-1}$$
$$= (A \,.\, A^{m-2})^{-1} \,.\, A^{-1} = [(A^{m-2})^{-1} \,.\, A^{-1}] \,.\, A^{-1} = (A^{m-2})^{-1} (A^{-1})^2$$
$$= (A \,.\, A^{m-3})^{-1} \,.\, (A^{-1})^2 = [(A^{m-3})^{-1} \,.\, A^{-1}] (A^{-1})^2 = (A^{m-3})^{-1} (A^{-1})^3$$
$$= A^{-1} (A^{-1})^{m-1} = (A^{-1})^m$$ **Proved.**

EXERCISE 31.3

Find the adjoint and inverse of the following matrices: (1 - 3)

1. $\begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$

4. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then show that $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$

5. If $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}, P = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$, show that $P^{-1} AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (***Ranchi University 2019***)

6. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}, B = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, show that $(AB)^{-1} = B^{-1} A^{-1}$.

7. Given the matrix $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix}$ compute det (A), A^{-1} and the matrix B such that $AB = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix}$

Also compute BA. Is $AB = BA$?

8. Find the condition of k such that the matrix

$A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & k & 6 \\ -1 & 5 & 1 \end{bmatrix}$ has an inverse. Obtain A^{-1} for $k = 1$.

9. Prove that $(A^{-1})^T = (A^T)^{-1}$.

10. If $A\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is

(*a*) $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ (*b*) $\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ (*c*) $\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ (*d*) $\begin{bmatrix} 2 & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ (***AMIETE, June 2010***)

ANSWERS

Find the adjoint and inverse of the following matrices: (1 – 3)

1. $\frac{1}{4}\begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix}$ **2.** $-\frac{1}{3}\begin{bmatrix} 6 & 6 & -15 \\ 1 & 0 & -1 \\ -5 & -3 & 8 \end{bmatrix}$ **3.** $\frac{1}{20}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$

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7. $5, \frac{1}{5}\begin{bmatrix} 9 & -2 & -4 \\ 1 & 2 & -1 \\ -12 & 1 & 7 \end{bmatrix}. B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, AB \neq BA$

8. $k \neq -\frac{3}{5}, A^{-1} = \frac{1}{8}\begin{bmatrix} -29 & 17 & 14 \\ -9 & 5 & 6 \\ 16 & -8 & -8 \end{bmatrix}$ **10.** (*d*)

31.12 ELEMENTARY TRANSFORMATIONS

Any one of the following operations on a matrix is called an elementary transformation.

1. Interchanging any two rows (or columns). This transformation is indicated by R_{ij}, if the *i*th and *j*th rows are interchanged.
2. Multiplication of the elements of any row R_i (or column) by a non-zero scalar quantity k is denoted by $(k.R_i)$.
3. Addition of constant multiplication of the elements of any row R_j to the corresponding elements of any other row R_j is denoted by $(R_i + kR_j)$.

If a matrix B is obtained from a matrix A by one or more E-operations, then B is said to be equivalent to A. The symbol ~ is used for equivalence.

$$i.e., A \sim B.$$

Example 1. *Reduce the following matrix to upper triangular form (Echelon form) :*

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix}$$

Solution. *Upper triangular matrix*. If in a square matrix, all the elements below the principal diagonal are zero, the matrix is called an upper triangular matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -7 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} R_3 \to R_3 + 5R_2$$ **Ans.**

Example 2. *Transform* $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix}$ *into a unit matrix.*

Solution. $$\begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & 4 \\ 0 & -1 & -5 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & -5 \end{bmatrix} R_2 \to -\frac{1}{2}R_2 \sim \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -2 \\ 0 & 0 & -7 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 3R_2 \\ \\ R_3 \to R_3 + R_2 \end{matrix}$$

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$$\sim\begin{bmatrix}1 & 0 & 9\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix} R_3 \to -\frac{1}{7}R_3 \quad \sim\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \begin{matrix}R_1 \to R_1 - 9R_3\\R_2 \to R_2 + 2R_3\\ \end{matrix}$$ **Ans.**

TO COMPUTE THE INVERSE OF A MATRIX FROM ELEMENTARY

MATRICES (Gauss-jordan Method)

If A is reduced to I by elementary transformation then

$$PA = I \qquad \text{where} \qquad P = P_n P_{n-1} \ldots P_2 P_1$$

$$\therefore \qquad P = A^{-1} \qquad\qquad = \text{Elementary matrix.}$$

Working rule. Write $A = IA$. Perform elementary row transformation on A of the left side and on I of the right hand side so that A is reduced to I and I of right hand side is reduced to P getting $I = PA$.

Then P is the inverse of A.

31.13 THE INVERSE OF A SYMMETRIC MATRIX

The elementary transformations are to be transformed so that the property of being symmetric is preserved. This requires that the transformations occur in pairs, a row transformation must be followed immediately by the same column transformation.

Example 1. *Find the inverse of the following matrix employing elementary transformations*:

$$\begin{bmatrix}3 & -3 & 4\\2 & -3 & 4\\0 & -1 & 1\end{bmatrix}$$ **(*U.P. I Semester Compartment 2013*)**

Solution. The given matrix is $A = \begin{bmatrix}3 & -3 & 4\\2 & -3 & 4\\0 & -1 & 1\end{bmatrix}$

$$\begin{bmatrix}3 & -3 & 4\\2 & -3 & 4\\0 & -1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} A \quad \Rightarrow \begin{bmatrix}1 & -1 & \frac{4}{3}\\2 & -3 & 4\\0 & -1 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} A \quad R_1 \to \frac{R_1}{3}$$

$$\Rightarrow \begin{bmatrix}1 & -1 & \frac{4}{3}\\0 & -1 & \frac{4}{3}\\0 & -1 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & 0 & 0\\-\frac{2}{3} & 1 & 0\\0 & 0 & 1\end{bmatrix} A \quad R_2 \to R_2 - 2R_1 \qquad \Rightarrow \begin{bmatrix}1 & -1 & \frac{4}{3}\\0 & 1 & -\frac{4}{3}\\0 & -1 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & 0 & 0\\\frac{2}{3} & -1 & 0\\0 & 0 & 1\end{bmatrix} A \quad R_2 \to -R_2$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & -1 & 0 \\ \frac{2}{3} & -1 & 1 \end{bmatrix} A \quad R_3 \to R_3 + R_2$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & -1 & 0 \\ -2 & 3 & -3 \end{bmatrix} A \quad R_3 \to -3\,R_3$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A \quad \begin{matrix} R_1 \to R_1 - \frac{4}{3} R_3 \\ R_2 \to R_2 + \frac{4}{3} R_3 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A \quad R_1 \to R_1 + R_2$$

Hence, $$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$ **Ans.**

EXERCISE 31.4

Reduce the matrices to triangular form:

1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix}$ **2.** $\begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & -5 \\ 0 & 1 & 5 \end{bmatrix}$

Find the inverse of the following matrices:

3. $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ **4.** $\begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix}$

Use elementary row operations to find inverse of

5. $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ (*AMIETE, June 2010*) **6.** $\begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & -3 \\ -1 & 2 & 1 & -1 \\ 2 & -3 & -1 & 4 \end{bmatrix}$

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7. $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & -3 \\ -1 & 2 & 1 & -1 \\ 2 & -3 & -1 & 4 \end{bmatrix}$

9. $\begin{bmatrix} 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ -1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 3 & 3 & 2 & 1 \\ 1 & 4 & 3 & 3 & -1 \\ 1 & 3 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & -2 & -1 & 2 & 2 \end{bmatrix}$

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1. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$

2. $\begin{bmatrix} 0 & 1 & 4 \\ 0 & 5 & -19 \\ 0 & 0 & 22 \end{bmatrix}$

3. $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -9 & 5 \end{bmatrix}$

5. $\frac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$

6. $\frac{1}{18}\begin{bmatrix} 2 & 5 & -7 & 1 \\ 5 & -1 & 5 & -2 \\ -7 & 5 & 11 & 10 \\ 1 & -2 & 10 & 5 \end{bmatrix}$

7. $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & -2 & 2 & -3 \\ 0 & 1 & -1 & 1 \\ -2 & 3 & -2 & 3 \end{bmatrix}$

8. $\frac{1}{18}\begin{bmatrix} 2 & 5 & -7 & 1 \\ 5 & -1 & 5 & -2 \\ -7 & 5 & 11 & 10 \\ 1 & -2 & 10 & 5 \end{bmatrix}$

9. $\begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix}$

10. $\frac{1}{15}\begin{bmatrix} 30 & -20 & -15 & 25 & -5 \\ 30 & -11 & -18 & 7 & -8 \\ -30 & 12 & 21 & -9 & 6 \\ -15 & 12 & 6 & -9 & 6 \\ 15 & -7 & -6 & -1 & -1 \end{bmatrix}$

31.14 RANK OF A MATRIX

The rank of a matrix is said to be r if

(*a*) It has at least one non-zero minor of order r.

(*b*) Every minor of A of order higher than r is zero.

Notes:(*i*) Non-zero row is that row in which all the elements are not zero.

(*ii*) The rank of the product matrix AB of two matrices A and B is less than the rank of either of the matrices A and B.

(*iii*) Corresponding to every matrix A of rank r, there exist non-singular matrices P and Q such that $$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

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31.15 NORMAL FORM (CANONICAL FORM)

By performing elementary transformation, any non-zero matrix A can be reduced to one of the following four forms, called the Normal form of A :

(*i*) I_r (*ii*) $[I_r\ 0]$ (*iii*) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (*iv*) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

The number r so obtained is called the rank of A and we write $\rho(A) = r$. The form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is called first canonical form of A. Since both row and column transformations may be used here, the element 1 of the first row obtained can be moved in the first column. Then both the first row and first column can be cleared of other non-zero elements. Similarly, the element 1 of the second row can be brought into the second column, and so on.

Example 1. *Find the rank of the following matrix by reducing it to normal form*

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} R_3 \to R_3 - R_2$$

$$C_2 \to C_2 - 2C_1,\ C_3 \to C_3 + C_1,\ C_4 \to C_4 - 3C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & 11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & 11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4 \to R_4 + \frac{1}{2} R_3$$

$$C_3 \to C_3 + \frac{6}{7} C_2, \quad C_4 \to C_4 - \frac{11}{7} C_2,$$

$$C_4 \to C_4 + 2C_3 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to -1/7\, R_2 \\ R_3 \to -1/2\, R_3 \end{matrix}$$

Rank of $A = 3$ **Ans.**

Example 2. *For which value of 'b' the rank of the matrix*

$$A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix} \text{ is 2,}$$ **(*U.P. I Semester 2016*)**

Solution. Here, we have

$$A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 0 & 13-5b & 10-4b \end{bmatrix} R_3 \to R_3 - bR_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 13-5b & 10-4b \end{bmatrix} \begin{matrix} C_2 \to C_2 - 5C_1 \\ C_3 \to C_3 - 4C_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & \frac{2(2-b)}{3} \end{bmatrix} R_3 \to R_3 - \frac{13-5b}{3} R_2$$

If rank of A is 2, then $\frac{2(2-b)}{3}$ must be zero.

i.e; $\frac{2(2-b)}{3} = 0 \quad \Rightarrow 2 - b = 0 \quad \Rightarrow b = 2$ **Ans.**

31.16 RANK OF MATRIX BY TRIANGULAR FORM

Rank = Number of non-zero row in upper triangular matrix.

Note. Non-zero row is that row which does not contain all the elements as zero.

Example 1. *Find the rank of the matrix*

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + R_2$$

Rank = Number of non zero rows = 2. **Ans.**

Example 2. *Find the rank of the matrix*

$$\begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$$

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Solution. $\begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & -2 & 14 & -4 \\ 0 & -2 & 14 & -4 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + 3R_1 \\ R_4 \to R_4 + 5R_1 \end{matrix}$

$$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 2R_2 \end{matrix}$$

Here the 4th order and 3rd order minors are zero. But a minor of second order

$$\begin{vmatrix} 3 & -2 \\ 7 & -2 \end{vmatrix} = -6 + 14 = 8 \neq 0$$

Rank = Number of non-zero rows = 2. **Ans.**

Example 3. *Use elementary transformation to reduce the following matrix A to triangular from and hence find the rank of A.*

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

(*R.G.P.V. Bhopal June 2018, U.P. I Semester Dec. 2018*)

Solution. We have,

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \approx \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} R_1 \longleftrightarrow R_2$$

$$\approx \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{matrix} \approx \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 33/5 & 22/5 \end{bmatrix} \begin{matrix} R_3 \to R_3 - 4/5\, R_2 \\ R_4 \to R_4 - 9/5\, R_2 \end{matrix}$$

$$\approx \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4 \to R_4 - R_3$$

$R(A)$ = Number of non-zero rows.

$\therefore \quad R(A) = 3$ **Ans.**

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EXERCISE 31.5

Find the rank of the following matrices:

1. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$ **3.** $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 2 & 4 & 3 & -2 \\ -3 & -2 & -1 & 4 \\ 6 & -1 & 7 & 2 \end{bmatrix}$ **5.** $\begin{bmatrix} 3 & 4 & 1 & 1 \\ 2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 1 & -1 & 2 & -3 \end{bmatrix}$ **6.** $\begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$

Reduce the following matrices to Echelon form and find out the rank

7. $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

9. $\begin{bmatrix} 3 & 2 & 5 & 7 & 12 \\ 1 & 1 & 2 & 3 & 5 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$ **10.** $\begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$ $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$, Rank = 3

Using elementary transformations, reduce the following matrices to the canonical form (or row-reduced Echelon form):

11. $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 & 1 \\ 0 & 3 & 4 & 1 & 2 \end{bmatrix}$ **12.** $A = \begin{bmatrix} 0 & 4 & -12 & 8 & 9 \\ 0 & 2 & -6 & 2 & 5 \\ 0 & 1 & -3 & 6 & 4 \\ 0 & -8 & 24 & 3 & 1 \end{bmatrix}$

Using elementary transformations, reduce the following matrices to the normal form:

13. $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$ **14.** $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 1 & 2 \end{bmatrix}$

Obtain a matrix N in the normal form equivalent to

15. $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 5 & 0 & 0 \\ 0 & 9 & 1 & -1 & 2 \\ 0 & 10 & 0 & 1 & 11 \end{bmatrix}$

Hence find non-singular matrices P and Q such that $PAQ = N$.

16. $\begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & -2 \end{bmatrix}$

Find the rank of the following matrix by reducing it into normal form:

17. $A = \begin{bmatrix} 1 & 3 & 2 & 5 & 1 \\ 2 & 2 & -1 & 6 & 3 \\ 1 & 1 & 2 & 3 & -1 \\ 0 & 2 & 5 & 2 & -3 \end{bmatrix}$ **18.** $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Choose the correct answer:

19. Rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ is

(*a*) 0 (*b*) 1 (*c*) 3 (*d*) 2 (***D.U. 2018***)

ANSWERS

1. 2 **2.** 3 **3.** 2

4. 3 **5.** 4 **6.** 2

7. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, Rank = 3 **8.** $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$, Rank = 3

9. $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$, Rank = 2 **10.** $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$, Rank = 3

13. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ **14.** $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

17. 3 **18.** 4

19. (*d*)

31.17 SOLUTION OF SIMULTANEOUS EQUATIONS

The matrix of the coefficients of *x, y, z* is reduced into Echelon form by elementary row transformations. At the end of the row transformation the value of *z* is calculated from the last equation and value of *y* and the value of *x* are calculated by the backward substitution.

Solve with the help of matrices, the simultaneous equations.

(***Vidyasagar Univercity 2018***)

Example 1. Solve:

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + 9z = 6$$

Solution. Above equations written in matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} R_3 \to R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$x + y + z = 3$ ...(1)

$y + 2z = 1$ ...(2)

$2z = 0 \Rightarrow z = 0$

putting $z = 0$ in (2), we get

$y = 1$

putting the value of y and z in (1), we get

$x = 2$

Hence $x = 2, y = 1, z = 0$ **Ans**

Example 2. *Find the general solution of the system of equations:*

$$3x_1 + 2x_3 + 2x_4 = 0$$
$$-x_1 + 7x_2 + 4x_3 + 9x_4 = 0$$
$$7x_1 - 7x_2 - 5x_4 = 0$$

Solution. The system of equations in the matrix form is expressed as

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ -1 & 7 & 4 & 9 \\ 7 & -7 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 7 & 4 & 9 \\ 3 & 0 & 2 & 2 \\ 7 & -7 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} -1 & 7 & 4 & 9 \\ 0 & 21 & 14 & 29 \\ 0 & 42 & 28 & 58 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 7R_1 \end{matrix}$$

$$\begin{bmatrix} -1 & 7 & 4 & 9 \\ 0 & 21 & 14 & 29 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} R_3 \to R_3 - 2R_2$$

$-x_1 + 7x_2 + 4x_3 + 9x_4 = 0$ ... (1)

$21x_2 + 14x_3 + 29x_4 = 0$ ... (2)

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Let $x_4 = a,\ x_3 = b$

From (2), $21\,x_2 + 14\,b + 29\,a = 0$ or $x_2 = -\frac{2b}{3} - \frac{29a}{21}$

From (1), $-x_1 + 7\left(-\frac{2b}{3} - \frac{29a}{21}\right) + 4b + 9a = 0$

$$x_1 = -\frac{2a}{3} - \frac{2b}{3}$$

$$x_1 = -\frac{2}{3}(a+b),\ x_2 = -\frac{1}{21}(29a + 14b)$$

$$x_3 = b,\ x_4 = a$$ **Ans.**

31.18 TYPES OF LINEAR EQUATIONS

(1) Consistent. A system of equations is said to be *consistent,* if they have one or more solution *i.e.*

$x + 2y = 4$	$x + 2y = 4$
$3x + 2y = 2$	$3x + 6y = 12$
Unique solution	Infinite solution

(2) Inconsistent. If a system of equation has no solution, it is said to be inconsistent *i.e.*

$$x + 2y = 4$$
$$3x + 6y = 5$$

31.19 CONSISTENCY OF A SYSTEM OF LINEAR EQUATIONS

$$a_{11}\,x_1 + a_{12}\,x_2 + \ldots a_{1n}\,x_n = b_1$$
$$a_{21}\,x_1 + a_{22}\,x_2 + \ldots a_{2n}\,x_n = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}\,x_1 + a_{m2}\,x_2 + \ldots a_{mn}\,x_n = b_m$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} \ldots\ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{bmatrix} \quad \text{and } C = [A, B] = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} & b_1 \\ a_{21} & a_{22} \ldots\ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} \ldots\ldots & a_{mn} & b_m \end{bmatrix}$$

$$\Rightarrow \quad AX = B$$

is called the **augmented** matrix.

$$\boxed{[A:B] = C}$$

(*a*) Consistent equations. If Rank A = Rank C

(*i*) *Unique solution:* Rank A = Rank $C = n$ [where n = number of unknown.]

(*ii*) *Infinite solution:* Rank A = Rank $C = r,\ r < n$

(*b*) Inconsistent equations. If Rank $A \neq$ Rank C.

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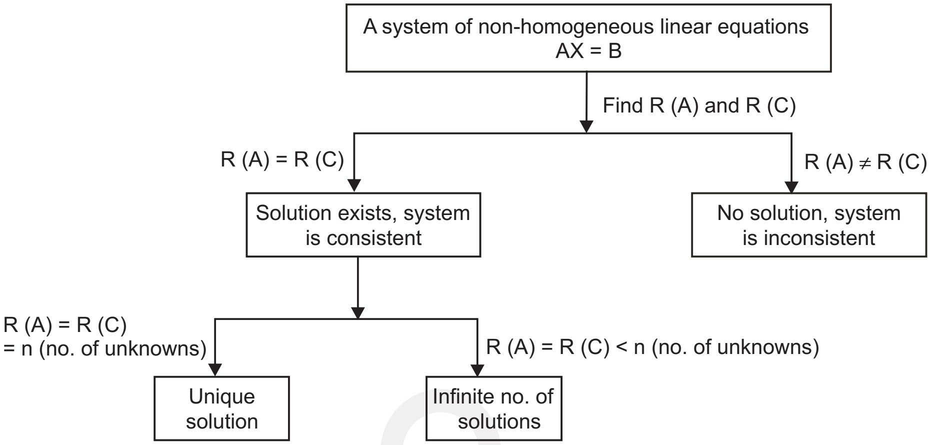
In Brief:



Example 1. *Show that the equations*

$$2x + 6y = -11,\ 6x + 20y - 6z = -3,\ 6y - 18z = -1$$

are not consistent.

Solution. Augmented matrix $C = [A, B]$

$$= \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 6 & 20 & -6 & : & -3 \\ 0 & 6 & -18 & : & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 6 & -18 & : & -1 \end{bmatrix} R_2 \to R_2 - 3R_1$$

$$\sim \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 0 & 0 & : & -91 \end{bmatrix} R_3 \to R_3 - 3R_2$$

The rank of C is 3 and the rank of A is 2.

Rank of $A \neq$ Rank of C. The equations are not consistent. **Ans.**

Example 2. *Test the consistency and hence solve the following set of equations.*

$$x_1 + 2x_2 + x_3 = 2$$

$$3x_1 + x_2 - 2x_3 = 1$$

$$4x_1 - 3x_2 - x_3 = 3$$

$$2x_1 + 4x_2 + 2x_3 = 4$$

Solution. The given set of equations is written in the matrix form:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & -3 & -1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

$$AX = B$$

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Here, we have augmented matrix $C = [A, B] \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 1 \\ 4 & -3 & -1 & 3 \\ 2 & 4 & 2 & 4 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & -5 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 4R_1 \\ R_4 \to R_4 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 \to -\frac{1}{5} R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + 11 R_2 \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to \frac{1}{6} R_3$$

Number of non-zero rows = Rank of matrix.

$$\Rightarrow \qquad R(C) = R(A) = 3$$

Hence, the given system is consistent and possesses a unique solution. In matrix form the system reduces to

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + x_3 = 2 \qquad \text{... (1)}$$
$$x_2 + x_3 = 1 \qquad \text{... (2)}$$
$$x_3 = 1$$

From (2), $\quad x_2 + 1 = 1 \quad \Rightarrow \quad x_2 = 0$

From (1), $\quad x_1 + 0 + 1 = 2 \quad \Rightarrow \quad x_1 = 1$

Hence, $\quad x_1 = 1, x_2 = 0$ and $x_3 = 1$ **Ans.**

Example 3. *Investigate the values of λ and μ so that the equations:*

$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$2x + 3y + \lambda z = \mu$$

have (i) no solution (ii) a unique solution

(iii) an infinite number of solutions. **(*R.G.P.V. Bhopal I Semester June 2007*)**

Solution. Here, we have,

$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$2x + 3y + \lambda z = \mu$$

The above equations are written in the matrix form

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

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$$AX = B$$

$$C = [A : B] = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 2 & 3 & \lambda & : & \mu \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & -\frac{15}{2} & -\frac{39}{2} & : & -\frac{47}{2} \\ 0 & 0 & \lambda - 5 & : & \mu - 9 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - \frac{7}{2} R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

(*i*) **No solution.** Rank $(A) \neq$ Rank (C)

$\lambda - 5 = 0$ or $\lambda = 5$ and $\mu - 9 \neq 0 \quad \Rightarrow \quad \mu \neq 9$

(*ii*) **A unique solution.** Rank $(A) = R\,(C)$ = Number of unknowns

$\lambda - 5 \neq 0 \quad \Rightarrow \quad \lambda \neq 5$ and $\mu \neq 9$

(*iii*) **An infinite number of solutions.** Rank (A) = Rank $(C) = 2$

$\lambda - 5 = 0$ and $\mu - 9 = 0$

$\lambda = 5$ and $\mu = 9$ **Ans.**

31.20 HOMOGENEOUS EQUATIONS

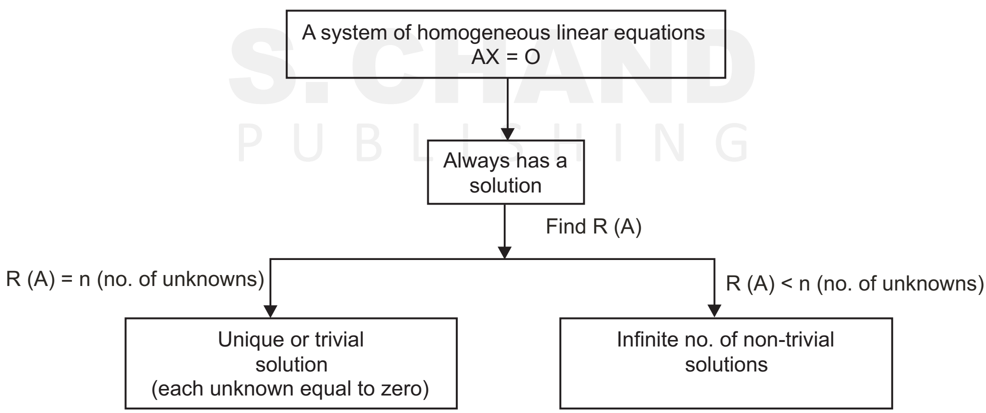
For a system of homogeneous linear equations $AX = O$

(*i*) $X = O$ is always a solution. This solution in which each unknown has the value zero is called the **Null Solution** or the **Trivial solution.** Thus a homogeneous system is always consistent.

A system of homogeneous linear equations has either the trivial solution or an infinite number of solutions.

(*ii*) If $R\,(A)$ = number of unknowns, the system has only the trivial solution.

(*iii*) If $R\,(A)$ < number of unknowns, the system has an infinite number of non-trivial solutions.



Example 1. *Determine 'b' such that the system of homogeneous equations*

$$2x + y + 2z = 0$$
$$x + y + 3z = 0$$
$$4x + 3y + bz = 0$$

has (i) Trivial solution

(ii) Non-Trivial solution. Find the Non-Trivial solution using matrix method.

(*U.P. I Sem Dec 2008*)

Solution. Here, we have

$$2x + y + 2z = 0$$

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$$x + y + 3z = 0$$

$$4x + 3y + bz = 0$$

(*i*) **For trivial solution:** We know that $x = 0$, $y = 0$ and $z = 0$. So, b can have any value.

(*ii*) **For non-trivial solution:** The given equations are written in the matrix form as :

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & b \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A\,X = B$$

$$R_1 \leftrightarrow R_2, \qquad R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 4R_1, \qquad R_3 \to R_3 - R_2$$

$$C = \begin{bmatrix} 2 & 1 & 2 & : 0 \\ 1 & 1 & 3 & : 0 \\ 4 & 3 & b & : 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & : 0 \\ 2 & 1 & 2 & : 0 \\ 4 & 3 & b & : 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & : 0 \\ 0 & -1 & -4 & : 0 \\ 0 & -1 & b-12 & : 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & : 0 \\ 0 & -1 & -4 & : 0 \\ 0 & 0 & b-8 & : 0 \end{bmatrix}$$

For non trivial solution or infinite solutions $R(C) = R(A) = 2 <$ Number of unknowns

$$b - 8 = 0, \quad b = 8$$ **Ans.**

31.21 CRAMER'S RULE

$$a_1 x + b_1 y + c_1 z = d_1$$

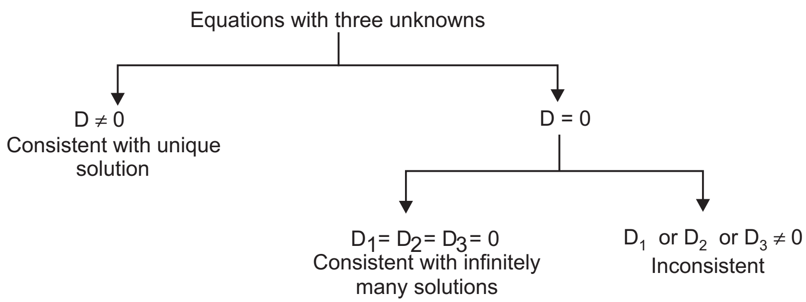
$$a_2 x + b_2 y + c_2 z = d_2$$

$$a_3 x + b_3 y + c_3 z = d_3$$

then $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\quad D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $\quad D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

$$x = \frac{D_1}{D}, \quad y = \frac{D_2}{D}, \quad z = \frac{D_3}{D}$$



EXERCISE 31.6

Test the consistency of the following equations and solve them if possible.

1. $3x + 3y + 2z = 1, \quad x + 2y = 4, \quad 10y + 3z = -2, \quad 2x - 3y - z = 5$

2. $x_1 - x_2 + x_3 - x_4 + x_5 = 1,\quad 2x_1 - x_2 + 3x_3 + 4x_5 = 2,$
$3x_1 - 2x_2 + 2x_3 + x_4 + x_5 = 1,\quad x_1 + x_3 + 2x_4 + x_5 = 0$

3. Find the value of k for which the following system of equations is consistent.
$3x_1 - 2x_2 + 2x_3 = 3,\ x_1 + kx_2 - 3x_3 = 0,\ 4x_1 + x_2 + 2x_3 = 7$

4. Find the value of λ for which the system of equations
$x + y + 4z = 1, x + 2y - 2z = 1,\ \lambda x + y + z = 1$
will have a unique solution.

5. Determine the values of a and b for which the system $\begin{bmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 3 \\ -1 \end{bmatrix}$

(*i*) has a unique solution, (*ii*) has no solution and, (*iii*) has infinitely many solutions.

6. Choose λ that makes the following system of linear equations consistent and find the general solution of the system for that λ.
$x + y - z + t = 2,\ 2y + 4z + 2t = 3,\ x + 2y + z + 2t = \lambda$

7. Show that the equations
$3x + 4y + 5z = a,\ 4x + 5y + 6z = b,\ 5x + 6y + 7z = c$
don't have a solution unless $a + c = 2b$.
Solve the equations when $a = b = c = -1$

8. Find the values of k, such that the system of equations
$4x_1 + 9x_2 + x_3 = 0,\ kx_1 + 3x_2 + kx_3 = 0,\ x_1 + 4x_2 + 2x_3 = 0$
has non-trivial solution. Hence, find the solution of the system.

9. Find values of λ for which the following system of equations has a non-trivial solution.
$3x_1 + x_2 - \lambda x_3 = 0,\ 2x_1 + 4x_2 + \lambda x_3 = 0,\ 8x_1 - 4x_2 - 6x_3 = 0$ **(*U.U. Odisha 2016*)**

10. Find value of λ so that the following system of homogeneous equations have exactly two linearly independent solutions
$\lambda x_1 - x_2 - x_3 = 0,\ -x_1 + \lambda x_2 - x_3 = 0,\ -x_1 - x_2 + \lambda x_3 = 0,$

11. Find the values of k for which the following system of equations has a non-trivial solution.
$(3k - 8)x + 3y + 3z = 0,\ 3x + (3k - 8)y + 3z = 0,\ 3x + 3y + (3k - 8)z = 0$
(*AMIETE June 2010*)

12. Solve the homogeneous system of equations :
$4x + 3y - z = 0,\ 3x + 4y + z = 0,\ x - y - 2z = 0,\ 5x + y - 4z = 0$

13. If $A = \begin{bmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 0 & 1 & \lambda \end{bmatrix}$

find the values of λ for which equation $AX = 0$ has (*i*) a unique solution, (*ii*) more than one solution.

14. Show that the following system of equations:
$x + 2y - 2u = 0,\ 2x - y - u = 0,\ x + 2z - u = 0,\ 4x - y + 3z - u = 0$
do not have a non-trivial solution.

15. Determine the values of λ and μ such that the following system has (*i*) no solution (*ii*) a unique solution (*iii*) infinite number of solutions:
$2x - 5y + 2z = 8,\ 2x + 4y + 6z = 5,\ x + 2y + \lambda z = \mu$

16. Test the following system of equations for consistency. If possible, solve for non-trivial solutions.
$3x + 4y - z - 6t = 0,\ 2x + 3y + 2z - 3t = 0,\ 2x + y - 14z - 9t = 0,\ x + 3y + 13z + 3t = 0$

17. Given the following system of equations
$2x - 2y + 5z + 3z = 0$, $4x - y + z + w = 0$, $3x - 2y + 3z + 4w = 0$, $x - 3y + 7z + 6w = 0$
Reduce the coefficient matrix A into Echelon form and find the rank utilising the property of rank, test the given system of equation for consistency and if possible find the solution of the given system.

18. Find the values of λ for which the equations
$(2 - \lambda) x + 2y + 3 = 0$, $2x + (4 - \lambda) y + 7 = 0$, $2x + 5y + (6 - \lambda) = 0$
are consistent and find the values of x and y corresponding to each of these values of λ.

ANSWERS

1. Consistent, $x = 2$, $y = 1$, $z = -4$

2. $x_1 = -3k_1 + k_2 - 1, x_2 = -3k_1 - 1, x_3 = k_1 - 2k_2 + 1, x_4 = k_1, x_4 = k_1, x_5 = k_2$

3. $k = \dfrac{1}{4}$

4. $\lambda \neq \dfrac{7}{10}$

5. (*i*) $a \neq -3$, (*ii*) $a = -3, b \neq \dfrac{1}{3}$, (*iii*) $a = -3, b = \dfrac{1}{3}$

6. $\lambda = \dfrac{7}{2}$, $x = \dfrac{1}{2} + 3k_2$, $y = \dfrac{3}{2} - 2k_2 - k_1$, $z = k_2$, $t = k_1$

7. $x = k + 1, y = -2k - 1, z = k$

8. $k = 1, x_1 = 2\lambda$, $x_2 = -\lambda, x_3 = \lambda$

9. $\lambda = 1$

10. $\lambda = -1$

11. $k = \dfrac{2}{3}, \dfrac{11}{3}$

12. $x = k, y = -k, z = k$

13. (*i*) $\lambda \neq 1$, (*ii*) $\lambda - 1$

15. (*i*) $\lambda = 3$, $\mu \neq \dfrac{5}{2}$ (*ii*) $\lambda \neq 3$, (*iii*) $\lambda = 3$, $\mu = \dfrac{5}{2}$

16. $x = 11k_1 + 6k_2, y = -8k_1 - 3k_2, z = k_1 \ \ t = k_2$

17. $x = 5k, y = 36k, z = 7k, w = 9k$

18. $\lambda = 1, -1, 12.$

32 CHAPTER

Eigen Values And Eigen Vectors

32.1 INTRODUCTION

Eigen values and eigen vectors are used in the study of ordinary differential equations, analysing population growth and finding powers of matrices.

32.2 EIGEN VALUES

Let
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$$

$$AX = Y \qquad ...(1)$$

Where A is the matrix, X is the column vector and Y is also column vector.

Here column vector X is transformed into the column vector Y by means of the square matrix A.

Let X be a such vector which transforms into λX by means of the transformation (1). Suppose the linear transformation $Y = AX$ transforms X into a scalar multiple of itself i.e. λX.

$$AX = Y = \lambda X$$

$$AX - \lambda IX = 0$$

$$(A - \lambda I) X = 0 \qquad ...(2)$$

Thus the unknown scalar λ is known as an eigen value of the matrix A and the corresponding non zero vector X as **eigen vector.**

The eigen values are also called characteristic values or proper values or latent values.

Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \text{ is characteristic matrix}$$

(*a*) **Characteristic Polynomial:** The determinant $|A - \lambda I|$ when expanded will give a polynomial, which we call as characteristic polynomial of matrix A.

For example; $\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$

$$= (2 - \lambda)(6 - 5\lambda + \lambda^2 - 2) - 2(2 - \lambda - 1) + 1(2 - 3 + \lambda)$$
$$= -\lambda^3 + 7\lambda^2 - 11\lambda + 5$$

(*b*) **Characteristic Equation:** The equation $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A *e.g.*

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

(*c*) **Characteristic Roots or Eigen Values:** The roots of characteristic equation $|A - \lambda I| = 0$ are called characteristic roots of matrix A. *e.g.*

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$
$$\Rightarrow \quad (\lambda - 1)(\lambda - 1)(\lambda - 5) = 0 \;\therefore\; \lambda = 1, 1, 5$$

Characteristic roots are 1, 1, 5.

Some Important Properties of Eigen Values (*Gawhati 2018, PTU. 2017*)

(1) Any square matrix A and its transpose A' have the same eigen values.

(2) The sum of the eigen values of a matrix is equal to the **trace** of the matrix.

(3) The product of the eigen values of a matrix A is equal to the **determinant** of A.

(4) If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A, then the eigen values of

(*i*) kA are $k\lambda_1, k\lambda_2, \ldots, k\lambda_n$ (*ii*) A^m are $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$

(*iii*) A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$.

Note. The sum of the elements on the principal diagonal of a matrix is called the **trace** of the matrix.

Example 1. *Find the characteristic roots of the matrix* $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Solution. The characteristic equation of the given matrix is (**M.U. 2018**)

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (6 - \lambda)(9 - 6\lambda + \lambda^2 - 1) + 2(-6 + 2\lambda + 2) + 2(2 - 6 + 2\lambda) = 0$$
$$\Rightarrow \quad -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

By trial, $\lambda = 2$ is a root of this equation.

$\Rightarrow \quad (\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0 \;\Rightarrow\; (\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$

$\Rightarrow \quad \lambda = 2, 2, 8$ are the characteristic roots or Eigen values. **Ans.**

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Example 2. *Find the eigen values of the matrix:*

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(*R.G.P.V. Bhopal, I Semester, June 2007*)

Solution. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

Expanding the determinant with the help of third row, we have

$\Rightarrow \quad (1-\lambda)\left[(2-\lambda)^2 - 1\right] = 0 \quad \Rightarrow \quad (1-\lambda)\left(\lambda^2 - 4\lambda + 4 - 1\right) = 0$

$\Rightarrow \quad (1-\lambda)\left(\lambda^2 - 4\lambda + 3\right) = 0 \quad \Rightarrow \quad (1-\lambda)(\lambda - 3)(\lambda - 1) = 0$

$\Rightarrow \quad \lambda = 1, 1, 3$

The eigen values of the given matrix are 1, 1 and 3. **Ans.**

Example 3. *The matrix A is defined as* $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Find the eigen values of $3A^3 + 5A^2 - 6A + 2I$.

Solution. $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (1 - \lambda)(3 - \lambda)(-2 - \lambda) = 0$ or $\lambda = 1, 3, -2$

Eigen values of $A^3 = 1, 27, -8$; Eigen values of $A^2 = 1, 9, 4$

Eigen values of $A = 1, 3, -2$; Eigen values of $I = 1, 1, 1$

$\therefore$ Eigen values of $3A^3 + 5A^2 - 6A + 2I$

First eigen value $= 3(1)^3 + 5(1)^3 - 6(1) + 2(1) = 4$

Second eigen value $= 3(27) + 5(9) - 6(3) + 2(1) = 110$

Third eigen value $= 3(-8) + 5(4) - 6(-2) + 2(1) = 10$

Required eigen values are 4, 110, 10 **Ans.**

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Example 4. *Show that for any square matrix A, the product of all the eigen values of A is equal to det (A), and the sum of all the eigen values of A is equal to the sum of the diagonal elements.*

Solution. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, |A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$

$|A - \lambda I|$

$= (a_{11} - \lambda)[(a_{22} - \lambda)(a_{33} - \lambda) - a_{32}a_{23}] - a_{12}[a_{21}(a_{33} - \lambda) - a_{31}a_{23}] + a_{13}[a_{21}a_{32} - a_{31}(a_{22} - \lambda)]$

$= (a_{11} - \lambda)[a_{22}a_{33} - (a_{22} + a_{33})\lambda + \lambda^2 - a_{32}a_{23}] - a_{12}[a_{21}a_{33} - a_{21}\lambda - a_{31}a_{23}] + a_{13}(a_{21}a_{32} - a_{31}a_{22} + a_{31}\lambda)$

$= a_{11}a_{22}a_{33} + (-a_{11}a_{22} - a_{11}a_{33})\lambda + a_{11}\lambda^2 - a_{11}a_{32}a_{23} + (-a_{22}a_{33} + a_{32}a_{23})\lambda + (a_{22} + a_{33})\lambda^2 - \lambda^3 - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{12}a_{21}\lambda + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} + a_{13}a_{31}\lambda$

$= -\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) + \lambda(-a_{11}a_{22} - a_{11}a_{33} + a_{12}a_{21} - a_{22}a_{33} + a_{23}a_{32} + a_{13}a_{31})$
$- [a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})] \quad \ldots(1)$

If $\lambda_1, \lambda_2, \lambda_3$ be the roots of the equation (1) then

Sum of the roots $= \lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$ = Sum of the diagonal elements.

Product of the roots

$= \lambda_1\lambda_2\lambda_3 = [a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})]$

= Determinant A. **Proved.**

Example 5. *Let λ be an eigen value of a matrix A. Then prove that*

(*i*) $\lambda + k$ is an eigen value of $A + kI$

(*ii*) $k\lambda$ is an eigen value of kA. **(*Gujarat II Semester June 2009*)**

Solution. Here, A has eigen value λ. $\Rightarrow |A - \lambda I| = 0 \quad \ldots (1)$

(*i*) Adding and subtracting kI from (1) we get

$$|A + kI - \lambda I - kI| = 0$$

$\Rightarrow \quad |(A + kI) - (\lambda + k)I| = 0 \Rightarrow \quad A + kI$ has $\lambda + k$ eigen value.

(*ii*) Multiplying (1), by k, we get

$k|A - \lambda I| = 0 \quad \Rightarrow \quad |kA - k\lambda I| = 0$

$\Rightarrow \quad kA$ has eigen value $k\lambda$. **Proved.**

Example 6. *If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A, find the eigen values of the martrix $(A - \lambda I)^2$.*

Solution. $(A - \lambda I)^2 = A^2 - 2\lambda AI + \lambda^2 I^2 = A^2 - 2\lambda A + \lambda^2 I$

Eigen values of A^2 are $\lambda_1^2, \lambda_2^2, \lambda_3^3 \ldots \lambda_n^2$

Eigen values of $2\lambda A$ are $2\lambda\lambda_1, \quad 2\lambda\lambda_2, \quad 2\lambda\lambda_3 \ldots 2\lambda\lambda_n$.

Eigen values of $\lambda^2 I$ are λ^2.

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$\therefore$ Eigen values of $A^2 - 2\lambda A + \lambda^2 I$

$$\lambda_1^2 - 2\lambda\lambda_1 + \lambda^2, \quad \lambda_2^2 - 2\lambda\lambda_2 + \lambda^2, \quad \lambda_3^2 - 2\lambda\lambda_3 + \lambda^2 \ldots\ldots$$

$$\Rightarrow \quad (\lambda_1 - \lambda)^2, \quad (\lambda_2 - \lambda)^2, \quad (\lambda_3 - \lambda)^2, \ldots (\lambda_n - \lambda)^2$$ **Ans.**

Example 7. *Prove that a matrix A and its transpose A′ have the same characteristic roots.*

Solution. Characteristic equation of matrix A is

$$| A - \lambda I | = 0 \qquad \ldots (1)$$

Characteristic equation of matrix A' is

$$| A' - \lambda I | = 0 \qquad \ldots (2)$$

Clearly both (1) and (2) are same, as we know that

$$| A | = | A' |$$

i.e., a determinant remains unchanged when rows be changed into columns and columns into rows. **Proved.**

Example 8. *If A and P be square matrices of the same type and if P be invertible, show that the matrices A and P^{-1} AP have the same characteristic roots.*

Solution. Let us put $B = P^{-1} AP$ and we will show that characteristic equations for both A and B are the same and hence they have the same characteristic roots.

$$B - \lambda I = P^{-1} AP - \lambda I = P^{-1} AP - P^{-1} \lambda IP = P^{-1} (A - \lambda I) P$$

$$\therefore \quad | B - \lambda I | = |P^{-1} (A - \lambda I) P | = | P^{-1} | \, |A - \lambda I| \, | P |$$

$$= |A - \lambda I | \, | P^{-1} | \, | P | = |A - \lambda I| \, | P^{-1}P|$$

$$= |A - \lambda I | \, | I | = | A - \lambda I| \text{ as } | I | = 1$$

Thus the matrices A and B have the same characteristic equations and hence the same characteristic roots. **Proved.**

Example 9. *If A and B be two square invertible matrices, then prove that AB and BA have the same characteristic roots.*

Solution. Now $AB = IAB = B^{-1} B (AB) = B^{-1} (BA) B$ ...(1)

But by Ex. 8, matrices BA and $B^{-1} (BA) B$ have same characteristic roots or matrices BA and AB by (1) have same characteristic roots. **Proved.**

Example 10. *If A and B be n rowed square matrices and if A be invertible, show that the matrices $A^{-1} B$ and BA^{-1} have the same characteristics roots.*

Solution. $A^{-1} B = A^{-1} BI = A^{-1} B (A^{-1}A) = A^{-1} (BA^{-1}) A.$ ...(1)

But by Ex. 8, matrices BA^{-1} and $A^{-1} (BA^{-1})A$ have same characteristic roots or matrices BA^{-1} and $A^{-1} B$ by (1) have same characteristic roots. **Proved.**

Example 11. *Show that 0 is a characteristic root of a matrix, if and only if, the matrix is singular.*

Solution. Characteristic equation of matrix A is given by

$$| A - \lambda I | = 0$$

If $\lambda = 0$, then from above it follows that $| A | = 0$ *i.e.* Matrix A is singular.

Again if Matrix A is singular *i.e.,* $| A | = 0$ then

$$| A - \lambda I | = 0 \;\Rightarrow\; | A | - \lambda | I | = 0, \; 0 - \lambda \cdot 1 = 0 \;\Rightarrow\; \lambda = 0.$$ **Proved.**

Example 12. *Show that characteristic roots of a triangular matrix are just the diagonal elements of the matrix.*

Solution. Let us consider the triangular matrix.

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Characteristic equation is $|A - \lambda I| = 0$

or
$$\begin{vmatrix} a_{11}-\lambda & 0 & 0 & 0 \\ a_{21} & a_{22}-\lambda & 0 & 0 \\ a_{31} & a_{32} & a_{33}-\lambda & 0 \\ a_{41} & a_{42} & a_{43} & a_{44}-\lambda \end{vmatrix} = 0$$

On expansion it gives

$$(a_{11} - \lambda)\,(a_{22} - \lambda)\,(a_{33} - \lambda)\,(a_{44} - \lambda) = 0$$

$\therefore$ $\lambda = a_{11},\quad a_{22},\quad a_{33},\quad a_{44}$

which are diagonal elements of matrix A. **Proved.**

Question *If λ is an eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also eigen value.*

[Try Yourself]

[**Hint:** $AA' = I$ if λ is the eigen value of A, then $\lambda^2 = 1, \quad \lambda = \frac{1}{\lambda}$]

Example 13. *Find the eigen values of the orthogonal matrix.*

$$B = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Solution. The characteristic equation of

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \text{ is } \begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(1-\lambda)\left[(1-\lambda)(1-\lambda)-4\right]-2\left[2(1-\lambda)+4\right]+2\left[-4-2(1-\lambda)\right]=0$

$\Rightarrow$ $(1-\lambda)\,(1 - 2\lambda + \lambda^2 - 4) - 2(2 - 2\lambda + 4) + 2\,(-4 - 2 + 2\lambda) = 0$

$\Rightarrow$ $\lambda^3 - 3\lambda^2 - 9\lambda + 27 = 0$ $\quad\Rightarrow\quad$ $(\lambda-3)^2(\lambda+3) = 0$

The eigen values of A are 3, 3, –3, so the eigen values $B = \frac{1}{3}A$ are 1, 1, –1.

Note. If $\lambda = 1$ is an eigen value of B then its reciproc $\frac{1}{\lambda} = \frac{1}{1} = 1$ is also an eigen value of B. **Ans.**

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EXERCISE 32.1

1. If λ be an eigen value of a non singular matrix A, show that $\frac{|A|}{\lambda}$ is an eigen value of matrix adj A.

2. There are infinitely many eigen vectors corresponding to a single eigen value.

3. Find the eigen values of the matrix $\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$

4. Find the eigen value $\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$

5. Find the product of the eigen value of the matrix $\begin{bmatrix} 3 & -3 & 3 \\ 2 & 1 & 1 \\ 1 & 5 & 6 \end{bmatrix}$

6. Find the sum of the eigen values of the matrix $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 4 & 1 & 5 \end{bmatrix}$

7. Find the eigen value of the inverse of the matrix $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$

8. Find the eigen value of the square of the matrix $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

9. Find the eigen values of the matrix $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}^3$

10. The sum and product of the eigen values of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are respectively

(*a*) 7 and 7 (*b*) 7 and 5 (*c*) 7 and 6 (*d*) 7 and 8 **(*AMIETE June 2010*)**

ANSWERS

3. Eigen values are 0, 1, –2	**4.** 1, 2, 5	**5.** 18
6. 11	**7.** $-1, \frac{1}{4}$	**8.** 1, 4, 9
9. 8, 27, 125	**10.** (*b*)	

32.3 CAYLEY-HAMILTON THEOREM

Satement. Every square matrix satisfies its own characteristic equation.

If $|A-\lambda I| = (-1)^n \left(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n\right)$ be the characteristic polynomial of $n \times n$ matrix $A = (a_{ij})$, then the matrix equation

$X^n + a_1 X^{n-1} + a_2 X^{n-2} + \ldots + a_n I = 0$ is satisfied by $X = A$ *i.e*

$$A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I = 0$$

Proof. Since the elements of the matrix $A - \lambda I$ are at most of the first degree in λ, the elements of adj. $(A - \lambda I)$ are at most degree $(n - 1)$ in λ. Thus, adj. $(A - \lambda I)$ may be written as a matrix polynomial in λ, given by

$$Adj\left(A-\lambda I\right) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots + B_{n-1}$$

where $B_0, B_1, \ldots, B_{n-1}$ are $n \times n$ matrices, their elements being polynomial in λ.

We know that $\left(A-\lambda I\right) Adj\left(A-\lambda I\right) = |A-\lambda I|\, I$

$$\left(A-\lambda I\right)\left(B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots + B_{n-1}\right) = (-1)^n\left(\lambda^n + a_1\lambda^{n-1} + \ldots + a_n\right) I$$

Equating coefficient of like power of λ on both sides, we get

$$-IB_0 = (-1)^n I$$

$$AB_0 - IB_1 = (-1)^n a_1 I$$

$$AB_1 - IB_2 = (-1)^n a_2 I$$

....

$$AB_{n-1} = (-1)^n a_n I$$

On multiplying the equation $A^n, A^{n-1}, \ldots, I$ respectively and adding, we obtain

$$0 = (-1)^n \left[A^n + a_1 A^{n-1} + \ldots + a_n I\right]$$

Thus $$A^n + a_1 A^{n-1} + \ldots + a_n I = 0$$

for example, Let A be square matrix and if $\lambda^3 - 2\lambda^2 + 3\lambda - 4 = 0$ ...(1)

be its characteristic equation, then according to Cayley Hamilton Theorem (1) is satisfied by A.

$$A^3 - 2A^2 + 3A - 4I = 0 \qquad \text{...(2)}$$

We can find can A^{-1} from equation (2). On premultiplying equation (2) by A^{-1}, we get

$$A^2 - 2A + 3I - 4A^{-1} = 0$$

$$A^{-1} = \frac{1}{4}[A^2 - 2A + 3I]$$

Example 1. *Verify Cayley-Hamilton theorem for the matrix*

(*Vidyasagar University 2018*)

$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ *and hence find* A^{-1}. **(*U.P.I Sem. Dec 2008*)**

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Solution. The characteristic equation of the matrix is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) - 4 = 0 \Rightarrow -1 + \lambda^2 - 4 = 0 \Rightarrow \lambda^2 - 5 = 0$$

By Cayley-Hamilton Theorem, $A^2 - 5I = 0$ ...(1)

Now, $A^2 = A.A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$$A^2 - 5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \quad \text{....(2)}$$

From (1) and (2), Cayley-Hamilton theorem is verified.

Again from (1), we have

$$A^2 - 5I = 0$$

Multiplying by A^{-1}, we get

$$A - 5A^{-1} = 0 \Rightarrow A^{-1} = \frac{1}{5}A \quad \Rightarrow \quad A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix} \textbf{ Ans.}$$

Example 2. *Verify Cayley-Hamilton Theorem for the following matrix:*

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

and use the theorem to find A^{-1}. **(*Delhi University April 2010*)**

Solution. We have $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Characteristic equation

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)(4 + \lambda^2 - 4\lambda - 1) + (\lambda - 2 + 1) + (1 + \lambda - 2) = 0$

$\Rightarrow \quad 2\lambda^2 - 8\lambda + 6 - \lambda^3 + 4\lambda^2 - 3\lambda + 2\lambda - 2 = 0$

$\Rightarrow \quad -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$

$\Rightarrow \quad \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

By Cayley-Hamilton theorem

$$A^3 - 6A^2 + 9A - 4I = 0 \quad \text{... (1)}$$

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$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

From equation (1), we get

$$\text{L.H.S.} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36+18-4 & -21+30-9 & 21-30+9 \\ -21+30-9 & 22-36+18-4 & -21+30-9 \\ 21-30+9 & -21+30-9 & 22-36+18-4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{R.H.S}$$

Verify Cayley-Hamilton Theorem.

From (1), $\quad A^3 - 6A^2 + 9A - 4I = 0$

$\Rightarrow \quad A^2 - 6A + 9I - 4A^{-1} = 0$

$$\Rightarrow \quad A^{-1} = \frac{1}{4}[A^2 - 6A + 9I]$$

$$= \frac{1}{4}\left[\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\right]$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Ans.

32.4 POWER OF MATRIX (by Cayley Hamilton Theorem)

Any positive integral power A^m of matrix A is linearly expressible in terms of those of lower degree, where m is a positive integer and n is the degree of characteristic equation such that $m > n$.

Example 1. *Find A^4 with the help of Cayley Hamilton Theorem, if*

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

Solution. Here, we have $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Characteristic equation of the matrix A is

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^3 - 6\lambda^2 - 11\lambda - 6 = 0$$

$$\Rightarrow \quad (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

Eigen values of A are 1, 2, 3.

Let $\lambda^4 = (\lambda^3 - 6\lambda^2 - 11\lambda - 6)Q(\lambda) + (a\lambda^2 + b\lambda + c) = 0$ ...(1)

(where Q (λ) is quotient)

Put $\lambda = 1$ in (1), $(1)^4 = a + b + c \quad \Rightarrow \quad a + b + c = 1$ ...(2)

Put $\lambda = 2$ in (1), $(2)^4 = 4a + 2b + c \quad \Rightarrow \quad 4a + 2b + c = 16$ ... (3)

Put $\lambda = 3$ in (1), $(3)^4 = 9a + 3b + c \quad \Rightarrow \quad 9a + 3b + c = 81$ ... (4)

Solving (2), (3) and (4), we get

$a = 25, \quad b = -60, \quad c = 36$

Replacing λ by matrix A in (1), we get

$$A^4 = (A^3 - 6A^2 + 11A - 6)Q(A) + (aA^2 + bA + c)$$

$$= O + aA^2 + bA + cI$$

$$= 25\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} + (-60)\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} + 36\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -25 & -50 & -100 \\ 125 & 150 & 100 \\ 250 & 250 & 225 \end{bmatrix} + \begin{bmatrix} -60 & 0 & 60 \\ -60 & -120 & -60 \\ -120 & -120 & -180 \end{bmatrix} + \begin{bmatrix} 36 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & 36 \end{bmatrix}$$

$$= \begin{bmatrix} -25-60+36 & -50+0+0 & -100+60+0 \\ 125-60+0 & 150-120+36 & 100-60+0 \\ 250-120+0 & 250-120+0 & 225-180+36 \end{bmatrix} = \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$$

Ans.

EXERCISE 32.2

1. Find the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Verify Cayley-Hamilton Theorem for this matrix. Hence find A^{-1}.

2. Use Cayley-Hamilton Theorem to find the inverse of the matrix

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

3. Using Cayley-Hamilton Theorem, find A^{-1}, given that

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 4 & -2 & 1 \end{bmatrix}$$

4. Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$ and show that the equation is also satisfied by A.

5. Using, Cayley-Hamilton Theorem obtain the inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

6. Show that the matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$

satisfies its characteristic equation. Hence find A^{-1}.

7. Verify Cayley-Hamilton Theorem for the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$ Hence evaluate A^{-1}.

8. Verify Cayley-Hamilton theorem for the matrix

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -2 \\ -1 & 1 & 2 \end{bmatrix}$$

9. Find adj. A by using Cayley-Hamilton thmeorem where A is given by

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$ ***(R.G.P.V., Bhopal, April 2010)***

10. If a matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, find the matrix A^{32}, using Cayley Hamilton Theorem.

ANSWERS

1. $A^{-1} = \dfrac{1}{20}\begin{bmatrix} 7 & -2 & -3 \\ 1 & 4 & 1 \\ -2 & 2 & 8 \end{bmatrix}$

2. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

3. $-\dfrac{1}{5}\begin{bmatrix} 4 & -5 & -2 \\ 7 & -10 & -1 \\ -2 & 0 & 1 \end{bmatrix}$

4. $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$.

5. $\dfrac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$

6. $\dfrac{1}{9}\begin{bmatrix} 7 & 2 & -10 \\ -2 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}$

7. $\frac{1}{11}\begin{bmatrix} -2 & 5 & -1 \\ -1 & -3 & 5 \\ 7 & -1 & -2 \end{bmatrix}$ **9.** $\begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 1 \\ -3 & 7 & 1 \end{bmatrix}$ **10.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 32 & 0 & 1 \end{bmatrix}$

32.5 CHARACTERISTIC VECTORS OR EIGEN VECTORS

A column vector X is transformed into column vector Y by means of a square matrix A. Now we want to multiply the column vector X by a scalar quantity λ so that we can find the same transformed column vector Y.

i.e., $AX = \lambda X$

X is known as eigen vector.

Example. *Show that the vector (1, 1, 2) is an eigen vector of the matrix*

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix} \quad \textit{corresponding to the eigen value 2.}$$

Solution. Let $X = (1, 1, 2)$.

Now, $$AX = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3+1-2 \\ 2+2-2 \\ 2+2+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 2X$$

Corresponding to each characteristic root λ, we have a corresponding non-zero vector X which satisfies the equation $[A - \lambda I]\, X = 0$. The non-zero vector X is called characteristic vector or Eigen vector.

32.6 PROPERTIES OF EIGEN VECTORS

(1) The eigen vector X of a matrix A is not unique.

(2) If $\lambda_1, \lambda_2, ..., \lambda_n$ be distinct eigen values of an $n \times n$ matrix then corresponding eigen vectors $X_1, X_2, ..., X_n$ form a linearly independent set.

(3) If two or more eigen values are equal it may or may not be possible to get linearly independent eigen vectors corresponding to the equal roots.

(4) Two eigen vectors X_1 and X_2 are called orthogonal vectors if $X_1' X_2 = 0$.

(5) Eigen vectors of a symmetric matrix corresponding to different eigen values are orthogonal.

Normalised form of vectors. To find normalised form of $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, we divide each element by $\sqrt{a^2 + b^2 + c^2}$.

For example, normalised form of $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$ $\left[\sqrt{1^2 + 2^2 + 2^2} = 3\right]$

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32.7 ORTHOGONAL VECTORS

Two vectors X and Y are said to be orthogonal if $X_1^T X_2 = X_2^T X_1 = 0$.

Example. *Determine whether the eigen vectors of the matrix*

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} \text{ are orthogonal.}$$

Solution. Characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (1-\lambda)[(2-\lambda)(3-\lambda)-2]-0-1[2-2(2-\lambda)]=0$

$\Rightarrow \quad (1-\lambda)(6-5\lambda+\lambda^2-2)-(2-4+2\lambda)=0 \quad \Rightarrow (\lambda-1)(\lambda^2-5\lambda+4)+2(\lambda-1)=0$

$\Rightarrow \quad (1-\lambda)(\lambda^2-5\lambda+4)-2(\lambda-1)=0 \quad \Rightarrow (\lambda-1)[\lambda^2-5\lambda+4+2]=0$

$\Rightarrow \quad (\lambda-1)(\lambda^2-5\lambda+6)=0 \quad \Rightarrow (\lambda-1)(\lambda-2)(\lambda-3)=0$

So, $\lambda = 1, 2, 3$ are three distinct eigen values of A.

For $\lambda = 1$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1-1 & 0 & -1 \\ 1 & 2-1 & 1 \\ 2 & 2 & 3-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_3 = 0$$

$$x_1 + x_2 + x_3 = 0 \Rightarrow x_2 = -x_3 - x_1$$

Let $x_1 = k$ then $x_2 = 0 - k = -k$

$$X_1 = \begin{bmatrix} k \\ -k \\ 0 \end{bmatrix} \quad \Rightarrow \quad X_1 = k \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

For $\lambda = 2$

$$\begin{bmatrix} 1-2 & 0 & -1 \\ 1 & 2-2 & 1 \\ 2 & 2 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 + 0x_2 + x_3 = 0$$

$$2x_1 + 2x_2 + x_3 = 0$$

$$\frac{x_1}{0-2} = \frac{x_2}{2-1} = \frac{x_3}{2-0} = k$$

$$\Rightarrow \quad x_1 = 2k, \quad x_2 = -k, \quad x_3 = -2k$$

$$X_2 = k\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

For λ = 3

$$\begin{bmatrix} 1-3 & 0 & -1 \\ 1 & 2-3 & 1 \\ 2 & 2 & 3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{matrix} -2x_1 + 0x_2 - x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \end{matrix}\right\} \Rightarrow \frac{x_1}{0-1} = \frac{x_2}{-1+2} = \frac{x_3}{2-0} = k$$

$$\Rightarrow \quad x_1 = k, \quad x_2 = -k, \quad x_3 = -2k \quad X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ -k \\ -2k \end{bmatrix} = k\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

$$X'_1 X_2 = [1, -1, \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}] = 3, \quad X'_2 X_3 \; [2, -1, -2]\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = 7, = \quad X'_3 X_1 = [1, -1, -2]\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 2$$

Since $X_1^T X_2 = 3 \neq 0, \quad X_2^T X_3 = 7 \neq 0, \quad X_3^T X_1 = 2 \neq 0$

Thus, there are three distinct eigen vectors. So X_1, X_2, X_3 are not orthogonal eigen vectors.

32.8 NON-SYMMETRIC MATRICES WITH NON-REPEATED EIGEN VALUES

Example 1. *Show that if* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *be the eigen values of the matrix A, then* A^n *has the eigen values* $\lambda^n{}_1, \lambda^n{}_2, \ldots \lambda^n{}_n$.

Solution. Let λ be an eigen value of the matrix A.

Therefore, $$AX = \lambda X \qquad \ldots(1)$$

By premultiplying both sides of (1) by A^{n-1}, we get

$$A^{n-1}(AX) = A^{n-1}(\lambda X) \quad \Rightarrow \quad A^n X = \lambda(A^{n-1}X) \qquad \ldots(2)$$

But $\quad A^2 X = A(AX) = A(\lambda X)$

$$= \lambda(AX) = \lambda(\lambda X) = \lambda^2 X \qquad \text{(From } (AX = \lambda X))$$

$$A^3 X = A(A^2 X) = \lambda(\lambda^2 X) = \lambda^3 X$$

Similarly, $$A^4 X = \lambda^4 X$$

....

....

$$A^n X = \lambda^n X$$

$\Rightarrow \quad \lambda^n$ is an eigen value of A^n.

Hence, if $\lambda_1, \lambda_2, \ldots\ldots\lambda_n$ be the eigen values of A, then $\lambda_1^n, \lambda_2^n, \lambda_3^n, \ldots\ldots\ldots\ldots\lambda_n^n$ be the eigen values of A^n. **Proved.**

Example 2. *If λ be an eigen value of matrix A (non-zero matrix), show that λ^{-1} is an eigen value of A^{-1}.*

Solution. We hav λ, is an eigen value of matrix A.

$$AX = \lambda X \qquad \ldots (1)$$

where X is eigen vector

Premultiplying both sides of (1) by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}(\lambda X) \quad \Rightarrow \quad (A^{-1}A)X = \lambda(A^{-1}X)$$

$$\Rightarrow \quad IX = \lambda(A^{-1}X) \quad \Rightarrow \quad X = \lambda(A^{-1}X)$$

$$\Rightarrow \quad \frac{1}{\lambda}X = A^{-1}X \quad \Rightarrow \quad A^{-1}X = \lambda^{-1}X$$

Henc λ^{-1}, is an eigen value of A^{-1}. **Proved.**

Example 3. *Find the eigen value and corresponding eigen vectors of the matrix*

$$A = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}$$ **(*U.P.I Sem. Dec. 2008*)**

Solution. $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (-5-\lambda)(-2-\lambda) - 4 = 0$$

$$\Rightarrow \lambda^2 + 7\lambda + 10 - 4 = 0 \Rightarrow \lambda^2 + 7\lambda + 6 = 0$$

$$(\lambda + 1)(\lambda + 6) = 0 \quad \Rightarrow \lambda = -1, -6$$

The eigen values of the given matrix are –1 and –6.

(*i*) When λ = –1, the corresponding eigen vectors are given by

$$\begin{bmatrix} -5+1 & 2 \\ 2 & -2+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 2x_1 - x_2 = 0 \Rightarrow x_1 = \frac{1}{2}x_2$$

Let $x_1 = k$, then $x_2 = 2k$. Hence, eigen vector $X_1 = \begin{bmatrix} k \\ 2k \end{bmatrix}$

(*ii*) When λ = –6 , the corresponding eigen vectors are given by

$$\begin{bmatrix} -5+6 & 2 \\ 2 & -2+6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 + 2x_2 = 0 \qquad \Rightarrow x_1 = -2x_2$$

Let $x_1 = k_1$, then $x_2 = -\frac{1}{2}k_1$

Hence eigen vector $X_2 = \begin{bmatrix} k_1 \\ -\frac{k_1}{2} \end{bmatrix}$ or $\begin{bmatrix} 2k_1 \\ -k_1 \end{bmatrix}$

Hence eigen vectors are $\begin{bmatrix} k \\ 2k \end{bmatrix}$ and $\begin{bmatrix} 2k_1 \\ -k_1 \end{bmatrix}$ **Ans.**

Example 4. *Find the eigen values and eigen vectors of matrix* $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

(*Mumbai University 2018, AMIETE June 2010, 2009*)

Solution. $|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda)(5-\lambda)$

Hence the characteristic equation of matrix A is given by

$$|A - \lambda I| = 0 \quad \Rightarrow \quad (3-\lambda)(2-\lambda)(5-\lambda) = 0$$

$\therefore \quad \lambda = 2, 3, 5.$

Thus the eigen values of matrix A are 2, 3, 5.

The eigen vectors of the matrix A corresponding to the eigen value λ is given by the non-zero solution of the equati $(A - \lambda I)\, X = 0$.

or $$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{... (1)}$$

When $\lambda = 2$, the corresponding eigen vector is given by

$$\begin{bmatrix} 3-2 & 1 & 4 \\ 0 & 2-2 & 6 \\ 0 & 0 & 5-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 + 4x_3 = 0$$

$$\Rightarrow \quad 0x_1 + 0x_2 + 6x_3 = 0$$

$$\frac{x_1}{6-0} = \frac{x_2}{0-6} = \frac{x_3}{0-0} = k \quad \Rightarrow \quad \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0} = k \quad \Rightarrow \quad x_1 = k,\; x_2 = -k,\; x_3 = 0$$

Hence $X_1 = \begin{bmatrix} k \\ -k \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ can be taken as an eigen vector of A corresponding to the eigen value $\lambda = 2$

When $\lambda = 3$, substituting in (1), the corresponding eigen vector is given by

$$\begin{bmatrix} 3-3 & 1 & 4 \\ 0 & 2-3 & 6 \\ 0 & 0 & 5-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x_1 + x_2 + 4x_3 = 0$$

$$0x_1 - x_2 + 6x_3 = 0$$

$$\frac{x_1}{6+4} = \frac{x_2}{0-0} = \frac{x_3}{0-0} \quad \Rightarrow \quad \frac{x_1}{10} = \frac{x_2}{0} = \frac{x_3}{0} = \frac{k}{10}$$

$$x_1 = k, x_2 = 0, \quad x_3 = 0$$

Hence, $X_2 = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ can be taken as an eigen vector of A corresponding to the eigen value $\lambda = 3$.

When $\lambda = 5$.

Again, when $\lambda = 5$, substituting in (1), the corresponding eigen vector is given by

$$\begin{bmatrix} 3-5 & 1 & 4 \\ 0 & 2-5 & 6 \\ 0 & 0 & 5-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + x_2 + 4x_3 = 0$$

$$-3x_2 + 6x_3 = 0$$

By cross-multiplication method, we have

$$\frac{x_1}{6+12} = \frac{x_2}{0+12} = \frac{x_3}{6-0} \quad \Rightarrow \quad \frac{x_1}{18} = \frac{x_2}{12} = \frac{x_3}{6} \quad \Rightarrow \quad \frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1} = k$$

$$x_1 = 3k, \quad x_2 = 2k, \quad x_3 = k$$

Hence, $X_3 = \begin{bmatrix} 3k \\ 2k \\ k \end{bmatrix} = k\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ can be taken as an eigen vector of A corresponding to the eigen value $\lambda = 5$. **Ans.**

EXERCISE 32.3

Non-symmetric matrix with different eigen values:

Find the eigen values and the corresponding eigen vectors for the following matrices:

1. $\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ **2.** $\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$.

3. $\begin{bmatrix} -9 & 2 & 6 \\ 5 & 0 & -3 \\ -16 & 4 & 11 \end{bmatrix}$ **4.** $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$

ANSWERS

1. $1, 2, 5; \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ **2.** $-2, 1, 3; \begin{bmatrix} 11 \\ 1 \\ 14 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

3. $-1, 1, 2; \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ **4.** $-1, 1, 4; \begin{bmatrix} -6 \\ -2 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

32.9 NON-SYMMETRIC MATRIX WITH REPEATED EIGEN VALUES

Example. *Find the eigen values and eigen vectors of the matrix:*

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution. We have, $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

On expanding the determinant by the third row, we get

$\Rightarrow \quad (1-\lambda)\{(2-\lambda)(2-\lambda)-1\} = 0 \quad \Rightarrow \quad (1-\lambda)\{(2-\lambda)^2 - 1\} = 0$

$\Rightarrow \quad (1-\lambda)(2-\lambda+1)(2-\lambda-1) = 0 \quad \Rightarrow \quad (1-\lambda)(3-\lambda)(1-\lambda) = 0$

$\Rightarrow \quad \lambda = 1, 1, 3$

when $\lambda = 1$

$$\begin{bmatrix} 2-1 & 1 & 1 \\ 1 & 2-1 & 1 \\ 0 & 0 & 1-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} R_2 \to R_2 - R_1 \quad \Rightarrow \quad x + y + z = 0$$

Let $x = k_1$ and $y = k_2$

$k_1 + k_2 + z = 0 \quad \Rightarrow \quad z = -(k_1 + k_2)$

$$X_1 = \begin{bmatrix} k_1 \\ k_2 \\ -(k_1 + k_2) \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \qquad [\text{If } k_1 = k_2 = k]$$

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Again $\lambda = 1$, $X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ [Again if $k_1 = 1$, $k_2 = 0$, $-(k_1 + k_2) = -1$]

when $\lambda = 3$

$$\begin{bmatrix} 2-3 & 1 & 1 \\ 1 & 2-3 & 1 \\ 0 & 0 & 1-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2 \to R_2 + R_1$$

$$-x + y + z = 0$$

$$2z = 0 \Rightarrow z = 0$$

$$-x + y + 0 = 0 \Rightarrow x = y = k \text{ (say)}$$

$$X_3 = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$ **Ans.**

EXERCISE 32.4

Non-symmetric matrices with repeated eigen values:

Find the eigen values and eigen vectors of the following matrices:

1. $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ **2.** $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ **3.** $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

4. $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ **5.** $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$ **6.** $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

ANSWERS

1. $-2, 2, 2$; $\begin{bmatrix} -4 \\ -1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ **2.** $1, 1, 5$; $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ **3.** $1, 1, 7$; $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

4. $-1, -1, 3$; $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ **5.** $1, 1, 1$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ **6.** $-2, -2, 4$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

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32.10 SYMMETRIC MATRICES WITH NON REPEATED EIGEN VALUES

Example 29. *Find the eigen values and the corresponding eigen vectors of the matrix*

$$\begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}$$

Solution. $|A - \lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & 5 & 4 \\ 5 & 7-\lambda & 5 \\ 4 & 5 & -2-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^3 - 3\lambda^2 - 90\lambda - 216 = 0$$

By trial: Take $\lambda = -3$, then $-27 - 27 + 270 - 216 = 0$

By synthetic division

–3	1	–3	–90	–216
		–3	18	216
	1	–6	–72	0

$$\lambda^2 - 6\lambda - 72 = 0 \quad \Rightarrow \quad (\lambda - 12)(\lambda + 6) = 0 \quad \Rightarrow \quad \lambda = -3, -6, \; 12$$

Matrix equation for eigen vectors $[A - \lambda I]\, X = 0$

$$\begin{bmatrix} -2-\lambda & 5 & 4 \\ 5 & 7-\lambda & 5 \\ 4 & 5 & -2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad ...(1)$$

Eigen Vector

On putting $\lambda = -3$ in (1), it will become

$$\begin{bmatrix} 1 & 5 & 4 \\ 5 & 10 & 5 \\ 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} x + 5y + 4z = 0 \\ 5x + 10y + 5z = 0 \end{cases}$$

$$\frac{x}{25-40} = \frac{y}{5-20} = \frac{z}{10-25} \qquad \text{or} \qquad \frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$$

Eigen vector $\quad X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Eigen vector corresponding to eigen value $\lambda = 6$.

Equation (1) becomes

$$\begin{bmatrix} 4 & 5 & 4 \\ 5 & 13 & 5 \\ 4 & 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{cases} 4x + 5y + 4z = 0 \\ 5x + 13y + 5z = 0 \end{cases}$$

$$\frac{x}{25-52} = \frac{y}{20-20} = \frac{z}{52-25} \qquad \text{or} \qquad \frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$$

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eigen vector $X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Eigen vector corresponding to eigen value $\lambda = 12$.

Equation (1) becomes

$$\begin{bmatrix} -14 & 5 & 4 \\ 5 & -5 & 5 \\ 4 & 5 & -14 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{cases} -14x + 5y + 4z = 0 \\ 5x - 5y + 5z = 0 \end{cases}$$

$$\frac{x}{25+20} = \frac{y}{20+70} = \frac{z}{70-25} \quad \text{or} \quad \frac{x}{1} = \frac{y}{2} = \frac{z}{1}$$

Eigen vector $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ **Ans.**

EXERCISE 32.5

Symmetric matrices with non-repeated eigen values:

Find the eigen values and eigen vectors of the following matrices:

1. $\begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$ **2.** $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ **3.** $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (*U.P., I Semester Jan 2011*)

4. $\begin{bmatrix} 2 & 4 & -6 \\ 4 & 2 & -6 \\ -6 & -6 & -15 \end{bmatrix}$ **5.** $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

ANSWERS

1. $-2, 4, 6; \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ **2.** $2, 3, 6; \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ **3.** $0, 3, 15; \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

4. $-2, 9, -18; \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ **5.** $-2, 3, 6; \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

32.11 SYMMETRIC MATRICES WITH REPEATED EIGEN VALUES

Example. *Find all the eigen values and eigen vectors of the matrix* $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

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Solution. The characteristic equation $\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$

$$\Rightarrow \quad (2-\lambda)[(2-\lambda)^2 - 1] + 1[-2+\lambda+1] + 1[1-2+\lambda] = 0$$

$$\Rightarrow \quad (2-\lambda)(4-4\lambda+\lambda^2-1) + (\lambda-1) + \lambda - 1 = 0$$

$$\Rightarrow \quad 8 - 8\lambda + 2\lambda^2 - 2 - 4\lambda + 4\lambda^2 - \lambda^3 + \lambda + 2\lambda - 2 = 0$$

$$\Rightarrow \quad -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$$

$$\Rightarrow \quad \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \qquad \text{... (1)}$$

On putting $\lambda = 1$ in (1), the equation (1) is satisfied. So $\lambda = -1$ is one factor of the equation (1). The other factor $(\lambda^2 - 5\lambda + 4)$ is got on dividing (1) $\lambda - 1$.

$$\Rightarrow \quad (\lambda-1)(\lambda^2 - 5\lambda + 4) = 0 \quad \text{or} \quad (\lambda-1)(\lambda-1)(\lambda-4) = 0 \quad \Rightarrow \quad \lambda = 1, 1, 4$$

The eigen values are 1, 1, 4.

When $\lambda = 4$ $\begin{pmatrix} 2-4 & -1 & 1 \\ -1 & 2-4 & -1 \\ 1 & -1 & 2-4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$-2x_1 - x_2 + x_3 = 0$$

$$x_1 - x_2 - 2x_3 = 0$$

$$\Rightarrow \quad \frac{x_1}{2+1} = \frac{x_2}{1-4} = \frac{x_3}{2+1} \qquad \Rightarrow \qquad \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1} = k$$

$$\Rightarrow \quad x_1 = k, \quad x_2 = -k, \quad x_3 = k$$

$$X_1 = \begin{bmatrix} k \\ -k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{or} \quad X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

When $\lambda = 1$ $\begin{pmatrix} 2-1 & -1 & 1 \\ -1 & 2-1 & -1 \\ 1 & -1 & 2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$\Rightarrow \quad \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0, \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$x_1 - x_2 + x_3 = 0$$

Let $x_1 = k_1$ and $x_2 = k_2$

$$k_1 - k_2 + x_3 = 0 \qquad \text{or} \qquad x_3 = k_2 - k_1$$

$$X_2 = \begin{bmatrix} k_1 \\ k_2 \\ k_2 - k_1 \end{bmatrix} \quad \Rightarrow \quad X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} k_1 = 1 \\ k_2 = 1 \end{bmatrix}$$

Let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

As X_3 is orthogonal to X_1 since the given matrix is symmetric

$$[1, -1, 1]\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \quad \text{or } l - m + n = 0 \qquad \text{... (2)}$$

As X_3 is orthogonal to X_2 since the given matrix is symmetric

$$[1, 1, 0]\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \quad \text{or } l + m + 0 = 0 \qquad \text{... (3)}$$

Solving (2) and (3), we get $\dfrac{l}{0-1} = \dfrac{m}{1-0} = \dfrac{n}{1+1} \Rightarrow \dfrac{l}{-1} = \dfrac{m}{1} = \dfrac{n}{2}$

$$X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$ **Ans.**

EXERCISE 32.6

Symmetric matrices with repeated eigen values

Find the eigen values and the corresponding eigen vectors of the following matrices:

1. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 6 & -3 & 3 \\ -3 & 6 & -3 \\ 3 & -3 & 6 \end{bmatrix}$

ANSWERS

1. $0, 0, 14; \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

2. $1, 3, 3; \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

3. $8, 2, 2; \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$

4. $3, 3, 12 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

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32.12 MATRIX HAVING ONLY ONE LINEARLY INDEPENDENT EIGEN VECTOR

Example. *Find the eigen values and eigen vectors of*

$$A = \begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$$

has less than three linearly independent eigen vectors. It is possible to obtain a similarity transformation that will diagonalise this matrix.

Solution. The characteristic equation of the given matrix is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} -3-\lambda & -7 & -5 \\ 2 & 4-\lambda & 3 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (-3-\lambda)[(4-\lambda)(2-\lambda)-6]+7[2(2-\lambda)-3]-5[4-(4-\lambda)]=0$$

$$\Rightarrow \quad \lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0 \quad \Rightarrow \quad (\lambda - 1)^3 = 1 \quad \Rightarrow \quad \lambda = 1, 1, 1$$

Eigen values of the given matrix A are 1, 1, 1. Eigen vector when $\lambda = 1$

$$\begin{bmatrix} -3-1 & -7 & -5 \\ 2 & 4-1 & 3 \\ 1 & 2 & 2-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad -4x_1 - 7x_2 - 5x_3 = 0 \qquad \ldots (1)$$

$$2x_1 + 3x_2 + 3x_3 = 0 \qquad \ldots (2)$$

$$\Rightarrow \quad \frac{x_1}{-21+15} = \frac{x_2}{-10+12} = \frac{x_3}{-12+14}$$

$$\Rightarrow \quad \frac{x_1}{-6} = \frac{x_2}{2} = \frac{x_3}{2} = k \text{ (say)}$$

Thus, $x_1 = -6k$, $x_2 = 2k$ and $x_3 = 2k$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6k \\ 2k \\ 2k \end{bmatrix} = 2k\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

All the eigen vectors are same and hence linearly independent. **Ans.**

32.13 MATRIX HAVING ONLY TWO EIGEN VECTORS

Example. *Find the eigen values and eigen vectors of* $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

has less than three linearly independent eigen vectors. Is it possible to obtain a similarity transformation that will diagonalise this matrix?

Solution. The characteristic equation of the given matrix A is

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$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (3-\lambda)[(-3-\lambda)(7-\lambda)+20]-10[-2(7-\lambda)+12]+5[-10-3(-3-\lambda)]=0$

$\Rightarrow \quad (3-\lambda)[-21+3\lambda-7\lambda+\lambda^2+20]-10[-14+2\lambda+12]+5[-10+9+3\lambda]=0$

$\Rightarrow \quad (3-\lambda)(\lambda^2-4\lambda-1)-10(2\lambda-2)+5(3\lambda-1)=0$

$\Rightarrow \quad \lambda^3-7\lambda^2+16\lambda-12=0 \quad \Rightarrow \quad (\lambda-3)(\lambda-2)(\lambda-2)=0 \quad \Rightarrow \quad \lambda = 3, 2, 2$

Eigen values of the given matrix A are 3, 2, 2.

Eigen vector, when $\lambda = 3$

$$\begin{bmatrix} 3-3 & 10 & 5 \\ -2 & -3-3 & -4 \\ 3 & 5 & 7-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 3 & 5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - 6x_2 - 4x_3 = 0 \quad \text{... (1)}$$

$$3x_1 + 5x_2 + 4x_3 = 0 \quad \text{... (2)}$$

Solving (1) and (2) by cross multiplication method, we have

$$\frac{x_1}{-24+20} = \frac{x_2}{-12+8} = \frac{x_3}{-10+18}$$

$$\Rightarrow \quad \frac{x_1}{-4} = \frac{x_2}{-4} = \frac{x_3}{8} = k \ \text{(say)}$$

Thus, $x_1 = -4k$, $x_2 = -4k$ and $x_3 = 8k$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4k \\ -4k \\ 8k \end{bmatrix} = 4k\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Eigen vector when $\lambda = 2$

$$\begin{bmatrix} 3-2 & 10 & 5 \\ -2 & -3-2 & -4 \\ 3 & 5 & 7-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 + 10x_2 + 5x_3 = 0 \quad \text{... (3)}$$

$$-2x_1 - 5x_2 - 4x_3 = 0 \quad \text{... (4)}$$

Solving (3) and (4) by cross multiplication method, we have

$$\frac{x_1}{-40+25} = \frac{x_2}{-10+4} = \frac{x_3}{-5+20} \quad \Rightarrow \quad \frac{x_1}{-15} = \frac{x_2}{-6} = \frac{x_3}{15} = k \ \text{(say)}$$

$\Rightarrow \quad x_1 = -15k, \quad x_2 = -6k, \quad x_3 = 15k$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -15k \\ -6k \\ 15k \end{bmatrix} = 3k\begin{bmatrix} -5 \\ -2 \\ 5 \end{bmatrix}$$

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We get one eigen vector corresponding to repeated root $\lambda_2 = 2 = \lambda_3$.

Eigen vectors corresponding to $\lambda_2 = 2 = \lambda_3$ are not linearly independent. Similarity transformation is not possible. **Ans.**

32.14 COMPLEX EIGEN VALUES

Example 1. *Show that if* $0 < \theta < \pi$, *then* $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ *has no real eigen values and consequently no eigen vector.* **(*Gujarat II Semester June 2009*)**

Solution. The characteristic equation of A is $\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$

$\Rightarrow \quad (\cos\theta - \lambda)^2 + \sin^2\theta = 0$

$\Rightarrow \quad \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta = 0$

$\Rightarrow \quad \lambda^2 - 2\lambda\cos\theta + 1 = 0$

$$\Rightarrow \quad \lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \frac{2\cos\theta \pm 2i\sqrt{1-\cos^2\theta}}{2} = \cos\theta \pm i\sin\theta$$

Hence, the given matrix A has no real eigen values and consequently no eigen vector. **Proved.**

Example 2. *If a matrix A is non-singular. Then* $\lambda = 0$ *is not its eigen value.*

Solution. Since matrix A is non-singular then $|A| \neq 0$

$$\Rightarrow \quad |A - 0I| \neq 0$$

Hence $\lambda = 0$ is not its eigen value. **Proved.**

32.15 ALGEBRAIC MULTIPLICITY

Algebraic multiplicity of an eigen value is the number of times of repetition of an eigen value.

It is denoted by $\text{mult}_a(\lambda)$.

For example, the eigen values of a matrix $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ are –3, –3, 5.

The $\text{mult}_a(-3) = 2$ and $\text{mult}_a(5) = 1$

32.16 GEOMETRIC MULTIPLICITY

Geometric multiplicity of an eigen value is the number of linearly independent eigen vectors corresponding to λ. It is denoted by $\text{Mult}_g(\lambda)$

In previous example two linearly independent eigen vectors corresponding to

$$\lambda = -3 \text{ are } \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

so the $\text{mult}_g(-3) = 2$

And the eigen vector corresponding to $\lambda = 5$ is $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ so the $\text{mult}_g(5) = 1$.

32.17 REGULAR EIGEN VALUE

If the algebraic multiplicity and geometric multiplicity of an eigen value are equal, then the eigen value is called *regular*.

Example. *Find the algebraic multiplicity and geometric multiplicity of an eigen value of the matrix* $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ *and show geometric multiplicity cannot be greater than algebraic multiplicity.*

Solution. The characteristic equation of the given matrix is

$$\begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \qquad \lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$$

$$\Rightarrow \qquad (\lambda - 2)(\lambda - 2)(\lambda - 3) = 0$$

$$\Rightarrow \qquad \lambda = 2, 2, 3$$

Therefore 2 is a multiple eigen value repeating 2 times. So Algebraic Multiplicity of 2 is 2.

$$\text{Mult}_a(2) = 2. \qquad \text{...(A)}$$

We shall find the eigen vector corresponding to the eigen value 2.

$$X = \begin{bmatrix} 3-2 & 10 & 5 \\ -2 & -3-2 & -4 \\ 3 & 5 & 7-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 10x_2 + 5x_3 = 0 \qquad \text{...(1)}$$

$$-2x_1 - 5x_2 - 4x_3 = 0 \qquad \text{...(2)}$$

Solving (1) and (2) by cross multiplication method, we have

$$\frac{x_1}{-40+25} = \frac{x_2}{-10+4} = \frac{x_3}{-5+20}$$

$$\Rightarrow \qquad \frac{x_1}{-15} = \frac{x_2}{-6} = \frac{x_3}{15} = k \quad \text{(say)}$$

Thus $x_1 = -15\,k$, $x_2 = -6\,k$, $x_3 = -15\,k$.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -15k \\ -6k \\ 15k \end{bmatrix} = 3k \begin{bmatrix} -5 \\ -2 \\ 5 \end{bmatrix}$$

Here the linearly independent eigen vector is 1.

So the, geometric multiplicity of eigen value 2 is 1

$$\text{Mult}_g(2) = 1 \qquad \text{...(B)}$$

Hence from (*A*) and (*B*) **Ans.**

Geometric multiplicity < Algebraic multiplicity

Notes: (1) If the values of x_1, x_2, x_3 are in terms of k (one independent value), then there is only one linearly independent eigen vector. So the geometric multiplicity is 1.

(2) If the values of x_1, x_2, x_3 are in terms of k_1, k_2 two independent values, then there are two linearly independent eigen vectors. So the geometric multiplicity is 2.

EXERCISE 32.7

From the following matrices; find eigen value, Algebraic multiplicity and Geometric multiplicity.

1. $\begin{bmatrix} -2 & -1 \\ 5 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

7. $\begin{bmatrix} 5 & 4 & -4 \\ 4 & 5 & -4 \\ -1 & -1 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (***M.U. 2018***)

ANSWERS

1. $\lambda = -1$, $\text{Mult}_a(-1) = 1$, $\text{Mult}_g(-1) = 1$

$\lambda = 3$, $\text{Mult}_a(3) = 1$, $\text{Mult}_g(3) = 1$

2. $\lambda = 1$, $\text{Mult}_a(1) = 3$, $\text{Mult}_g(1) = 1$

3. $\lambda = 1$, $\text{Mult}_a(1) = 3$, $\text{Mult}_g(1) = 1$

4. $\lambda = 2$, $\text{Mult}_a(2) = 2$, $\text{Mult}_g(2) = 1$

5. $\lambda = 5$, $\text{Mult}_a(5) = 1$, $\text{Mult}_g(5) = 1$

$\lambda = 2$, $\text{Mult}_a(2) = 2$, $\text{Mult}_g(2) = 1$

6. $\lambda = 1$, $\text{Mult}_a(1) = 2$, $\text{Mult}_g(1) = 1$

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7. $\lambda = 1$, $\text{Mult}_a\,(1) = 2$, $\text{Mult}_g\,(1) = 2$ $\lambda = 10$, $\text{Mult}_a\,(10) = 1$, $\text{Mult}_g\,(10) = 1$	**8.** $\lambda = 1$, $\text{Mult}_a(1) = 4$, $\text{Mult}_g(1) = 3$

32.18 SIMILARITY TRANSFORMATION

Let A and B be two square matrices of order n. Then B is said to be similar to A if there exists a non-singular matrix P such that

$$B = P^{-1}\,AP \qquad ...(1)$$

Equation (1) is called a similar transformation.

32.19 DIAGONALISATION OF A MATRIX

Diagonalisation of a matrix A is the process of reduction of A to a diagonal form 'D'. If A is related to D by a similarity transformation such that $D = P^{-1}\,AP$ then A is reduced to the diagonal matrix D through *modal matrix* P. D is also called *spectral matrix* of A.

32.20 ORTHOGONAL TRANSFORMATION OF A SYMMETRIC MATRIX TO DIAGONAL FORM

Let A be a symmetric matrix, then

$$A \cdot A' = \mathrm{I} \qquad ...(1)$$

and

$$A \cdot A^{-1} = I \qquad ...(2)$$

From (1) and (2), we have $A^{-1} = A'$

We know that, diagonalisation transformation of a symmetric matrix is

$$P^{-1}AP = D$$

If we normalize each eigen vector and use them to form the normalized modal matirx N then N is an orthogonal matrix.

Then,
$$N'\,AN = D$$

Transforming A into D by means of the transformation $N'\,AN = D$ is called as orthogonal transformation.

Note. To normalize eigen vector divide each element of the vector by the square root of the sum of the squares of all the elements of the vector.

Example. *Show that similar matrices have same trace.* **(*D.U. April 2010*)**

Solution. As we know that similar matrices have eigen value.

Trace of matrices is sum of all eigen value. Hence similar matrices have same trace. **Proved.**

32.21 THEOREM ON DIAGONALIZATION OF A MATRIX

Theorem. *If a square matrix A of order n has n linearly independent eigen vectors, then a matrix P can be found such that P^{-1} AP is a diagonal matrix.*

Proof. We shall prove the theorem for a matrix of order 3. The proof can be easily extended to matrices of higher order.

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Let
$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

and let $\lambda_1, \lambda_2, \lambda_3$ be its eigen values and X_1, X_2, X_3 the corresponding eigen vectors, where

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \qquad X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, \qquad X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

For the eigen value λ_1, the eigen vector is given by

$$\left.\begin{aligned} (a_1 - \lambda_1)x_1 + b_1 y_1 + c_1 z_1 = 0 \\ a_2 x_1 + (b_2 - \lambda_1) y_1 + c_2 z_1 = 0 \\ a_3 x_1 + b_3 y_1 + (c_3 - \lambda_1) z_1 = 0 \end{aligned}\right\} \quad ...(1)$$

$\therefore$ We have

$$\left.\begin{aligned} a_1 x_1 + b_1 y_1 + c_1 z_1 = \lambda_1 x_1 \\ a_2 x_1 + b_2 y_1 + c_2 z_1 = \lambda_1 y_1 \\ a_3 x_1 + b_3 y_1 + c_3 z_1 = \lambda_1 z_1 \end{aligned}\right\} \quad ...(2)$$

Similarly, for λ_2 and λ_3, we have

$$\left.\begin{aligned} a_1 x_2 + b_1 y_2 + c_1 z_2 = \lambda_2 x_2 \\ a_2 x_2 + b_2 y_2 + c_2 z_2 = \lambda_2 y_2 \\ a_3 x_2 + b_3 y_2 + c_3 z_2 = \lambda_2 z_2 \end{aligned}\right\} \quad ...(3)$$

and
$$\left.\begin{aligned} a_1 x_3 + b_1 y_3 + c_1 z_3 = \lambda_3 x_3 \\ a_2 x_3 + b_2 y_3 + c_2 z_3 = \lambda_3 y_3 \\ a_3 x_3 + b_3 y_3 + c_3 z_3 = \lambda_3 z_3 \end{aligned}\right\} \quad ...(4)$$

We consider the matrix
$$P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

whose columns are the eigen vectors of A.

Then
$$AP = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$= \begin{pmatrix} a_1 x_1 + b_1 y_1 + c_1 z_1 & a_1 x_2 + b_1 y_2 + c_1 z_2 & a_1 x_3 + b_1 y_3 + c_1 z_3 \\ a_2 x_1 + b_2 y_1 + c_2 z_1 & a_2 x_2 + b_2 y_2 + c_2 z_2 & a_2 x_3 + b_2 y_3 + c_2 z_3 \\ a_3 x_1 + b_3 y_1 + c_3 z_1 & a_3 x_2 + b_3 y_2 + c_3 z_2 & a_3 x_3 + b_3 y_3 + c_3 z_3 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{pmatrix} \quad \text{[Using results (2), (3) and (4)]}$$

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$$= \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = PD$$

where D is the Diagonal matrix $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$.

$$\therefore \qquad AP = PD$$

$$\Rightarrow \qquad P^{-1} AP = P^{-1} PD = D$$

Notes (1) The square matrix P, which diagonalises A, is found by grouping the eigen vectors of A into square-matrix and the resulting diagonal matrix has the eigen values of A as its diagonal elements.

(2) The transformation of a matrix A to $P^{-1} AP$ is known as a *similarity transformation*.

(3) The reduction of A to a diagonal matrix is, obviously, a particular case of similarity transformation.

(4) The matrix P which diagonalises A is called the *modal matrix* of A and the resulting diagonal matrix D is known as the *spectra matrix* of A.

Example. *Find the eigen values, eigen vectors the modal matrix and diagonalise the matrix given below.*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

Solution. The characteristic equation of the given matrix is

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)\{(3-\lambda)^2 - 1\} = 0 \quad \Rightarrow \quad (1-\lambda)(3-\lambda+1)(3-\lambda-1) = 0$$

$$\Rightarrow \quad (1-\lambda)(4-\lambda)(2-\lambda) = 0 \quad \Rightarrow \quad \lambda = 1, 2, 4$$

Eigen vectors

When $\lambda = 1$,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \to R_3 + \frac{1}{2} R_2$$

$$\Rightarrow \qquad 2x_2 - x_3 = 0 \qquad \qquad \text{... (1)}$$

$$\frac{3}{2} x_3 = 0 \Rightarrow x_3 = 0 \qquad \qquad \text{... (2)}$$

Putting $x_3 = 0$ from (2) in (1), we get $2x_2 - 0 = 0 \Rightarrow x_2 = 0$

$$\text{Eigen Vector} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

When λ = 2,

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} R_1 \to -R_1 \\ R_3 \to R_3 + R_2 \end{matrix}$$

$x_1 = 0$

$x_2 - x_3 = 0 \Rightarrow x_2 = x_3$, Eigen vector = $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

When λ = 4,

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x_1 = 0$

$-x_2 - x_3 = 0$

$x_2 = -x_3$

$$\text{Eigen Vector} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\text{Modal matrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$ **Ans.**

Let us diagonalise the given matrix:

$$P^{-1}AP = -\frac{1}{2}\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 2 & -4 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

EXERCISE 32.8

1. Find the matrix B which transforms the matrix

$$A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} \text{ to a diagonal matrix.}$$

2. For the matrix $A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$, determine a matrix ***P*** such that $P^{-1}AP$ is diagonal matrix.

3. Determine the eigen values and the corresponding eigen vectors of the matrix

$$A = \begin{bmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{bmatrix}$$

Hence find the matrix P such that $P^{-1}AP$ is diagonal matrix.

4. Reduce the following matrix A into a diagonal matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

ANSWERS

1. $B = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

2. $P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -\sqrt{2} & \sqrt{2} \\ 1 & 1 & 1 \end{bmatrix}$

3. $P = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

32.22 POWERS OF A MATRIX (By diagonalisation)

We can obtain powers of a matrix by using diagonalisation.

We know that $D = P^{-1} AP$

Where A is the square matrix and P is a non-singular matrix.

$$D^2 = (P^{-1} AP)(P^{-1} AP) = P^{-1} A (P P^{-1}) AP = P^{-1} A^2 P$$

Similarly $D^3 = P^{-1} A^3 P$

In general $D^n = P^{-1} A^n P$ ...(1)

Pre-multiply (1) by P and post-multiply by P^{-1}

$$P D^n P^{-1} = P (P^{-1} A^n P) P^{-1}$$
$$= (P P^{-1}) A^n (P P^{-1})$$
$$= A^n$$

Procedure: (1) Find eigen values for a square matrix A.

(2) Find eigen vectors to get the modal matrix P.

(3) Find the diagonal matrix D, by the formula $D = P^{-1}$ AP

(4) Obtain A^n by the formula $A^n = P D^n P^{-1}$.

Example. *Find a matrix P which transform the matrix* $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ *to diagonal form. Hence* A^4.

Solution. Characteristic equation of the matrix A is

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \quad \text{or} \quad (\lambda-1)(\lambda-2)(\lambda-3) = 0 \quad \Rightarrow \lambda = 1, 2, 3$$

For $\lambda = 1$, eigen vector is given by

$$\begin{bmatrix} 1-1 & 0 & -1 \\ 1 & 2-1 & 1 \\ 2 & 2 & 3-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 0x_1 + 0x_2 - x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}\right] \Rightarrow \frac{x_1}{0+1} = \frac{x_2}{-1+0} = \frac{x_3}{0} \text{ or } x_1 = 1, x_2 = -1, x_3 = 0$$

Eigen vector is [1, –1, 0].

For $\lambda = 2$, eigen vector is given by

$$\begin{bmatrix} 1-2 & 0 & -1 \\ 1 & 2-2 & 1 \\ 2 & 2 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_1 \to R_1 + R_2$$

$$\left.\begin{aligned} x_1 + 0x_2 + x_3 &= 0 \\ 2x_1 + 2x_2 + x_3 &= 0 \end{aligned}\right]$$

$$\Rightarrow \quad \frac{x_1}{0-2} = \frac{x_2}{2-1} = \frac{x_3}{2-0} \quad \Rightarrow \quad x_1 = -2, \quad x_2 = 1, \quad x_3 = 2$$

Eigen vector is [–2, 1, 2].

For $\lambda = 3$, eigen vector is given by

$$\begin{bmatrix} 1-3 & 0 & -1 \\ 1 & 2-3 & 1 \\ 2 & 2 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -2x_1 + 0x_2 - x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \end{aligned}\right]$$

$$\Rightarrow \quad \frac{x_1}{0-1} = \frac{x_2}{-1+2} = \frac{x_3}{2-0} \quad \Rightarrow \quad x_1 = -1, \quad x_2 = 1, \quad x_3 = 2$$

Eigen vector is [–1, 1, 2].

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Modal matrix $P = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$ and $P^{-1} = -\frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$

Now $\quad P^{-1}AP = \begin{bmatrix} 0 & -1 & \frac{1}{2} \\ -1 & -1 & 0 \\ 1 & 1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$

$$A^4 = PD^4P^{-1} = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}\begin{bmatrix} 0 & -1 & \frac{1}{2} \\ -1 & -1 & 0 \\ 1 & 1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$$ **Ans.**

EXERCISE 32.9

Find a matrix P which transforms the following matrices to diagonal form. Hence calculate the power matrix.

1. If $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$, calculate A^4.

2. If $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, calculate A^4.

3. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, calculate A^6.

4. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$, calculate A^8.

5. Show that the matrix A is diagonalisable $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$. If so obtain the matrix P such that P^{-1} AP is a diagonal matrix. **(*AMIETE June 2010*)**

ANSWERS

1. $\begin{bmatrix} 251 & 405 & 235 \\ 405 & 891 & 405 \\ 235 & 405 & 251 \end{bmatrix}$

2. $\begin{bmatrix} 251 & -405 & 235 \\ -405 & 891 & -405 \\ 235 & -405 & 251 \end{bmatrix}$

3. $\begin{bmatrix} 1366 & -1365 & 1365 \\ -1365 & 1366 & -1365 \\ 1365 & -1365 & 1366 \end{bmatrix}$

4. $\begin{bmatrix} -12099 & 12355 & 6305 \\ -12100 & 12356 & 6305 \\ -13120 & 13120 & 6561 \end{bmatrix}$

32.23 COMPLEX MATRICES

Conjugate of a Complex Number

$z = x + i\,y$ is called a complex number where $\sqrt{-1} = i$, x, y are real numbers. $\bar{z} = x - iy$ is called the conjugate of the complex number z, e.g.,

Complex number	Conjugate number
$2 + 3i$	$2 - 3i$
$-4 - 5i$	$-4 + 5i$
$6i$	$-6i$
2	2

Conjugate of a matrix. The matrix formed by replacing the elements of a matrix by their respective conjugate numbers is called the conjugate of A and is denoted by $\overline{A}$.

$$A = (a_{ij})_{m \times n}, \text{ then } \overline{A} = (\overline{a}_{ij})_{m \times n}$$

Example

If $A = \begin{bmatrix} 3+4i & 2-i & 4 \\ i & 2 & -3i \end{bmatrix}$ then $\overline{A} = \begin{bmatrix} 3-4i & 2+i & 4 \\ -i & 2 & 3i \end{bmatrix}$

THEOREM

If A and B be two matrices and their conjugate matrices are $\overline{A}$ and $\overline{B}$ respectively, then

(*i*) $(\overline{\overline{A}}) = A$ (*ii*) $(\overline{A+B}) = \overline{A} + \overline{B}$ (*iii*) $(\overline{kA}) = \overline{k}\,\overline{A}$ (*iv*) $(\overline{AB}) = \overline{A}\,\overline{B}$

Proof. Let $A = [a_{ij}]_{m \times n}$, then

$\overline{A} = [\overline{a}_{ij}]_{m \times n}$ where $\overline{a}_{ij}$ is the conjugate complex of a_{ij}.

The (i, j) th element of $(\overline{\overline{A}})$ = the conjugate complex of the (i, j)th element of $\overline{A}$

= the conjugate complex of $\overline{a}_{ij}$

= a_{ij} = the (i, j)th element of A.

Hence $(\overline{\overline{A}}) = A$. **Proved.**

(*ii*) Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$

$\overline{A} = [\overline{a}_{ij}]_{m \times n}$ and $\overline{B} = [\overline{b}_{ij}]_{m \times n}$

(i, j) th element of $(\overline{A+B})$ = conjugate complex of (i, j) th element of $(A + B)$

= conjugate complex of $(a_{ij} + b_{ij})$

$= (\overline{a_{ij} + b_{ij}}) = \overline{a}_{ij} + \overline{b}_{ij}$

= (i, j)th element of $\overline{A}$ + (i, j)th element of $\overline{B}$

= (i, j)th element of $(\overline{A} + \overline{B})$

Hence, $(\overline{A+B}) = \overline{A} + \overline{B}$ **Proved.**

(*iii*) Let $A = [a_{ij}]_{m \times n}$, let k be any complex number.

The (i, j)th element of $(\overline{kA})$ = conjugate complex of the (i, j)th element of kA

= conjugate complex of ka_{ij}

$= \overline{ka_{ij}} = \overline{k} \cdot \overline{a_{ij}}$

$= \overline{k} \cdot (i, j)$th element of $\overline{A} = (i, j)$th element of $\overline{k} . \cdot \overline{A}$

Hence, $\overline{kA} = \overline{k} \cdot \overline{A}$ **Proved.**

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(*iv*) Let $\qquad A = [a_{ij}]_{m \times n},\ B = [b_{ij}]_{n \times p}$

Then $\qquad \overline{A} = [\overline{a_{ij}}]_{m\times n},\quad \overline{B} = [\overline{b_{ij}}]_{n\times p}$

The (i, j)th element of $(\overline{AB})$ = conjugate complex of (i, j)th element of AB

$$= \text{conjugate complex of } \sum_{j=1}^{n} a_{ij}\, b_{jk} = \left(\sum_{j=1}^{n} \overline{a_{ij} b_{jk}}\right) = \sum_{j=1}^{n} \overline{a_{ij}} \cdot \overline{b_{jk}}$$

= (i, j)th element of $\overline{A} \cdot \overline{B}$

Hence, $\quad (\overline{AB}) = \overline{A} \cdot \overline{B}$ **Proved.**

32.24 TRANSPOSE OF CONJUGATE OF A MATRIX

The transpose of a conjugate of a matrix A is denoted by

$$(\overline{A})' = A^{\theta}$$

The (i, j)th element of A^{θ} = (j, i)th element of $\overline{A}$

= conjugate complex of (j, i)th element of A.

Example. *If* $A = \begin{bmatrix} 2+3i & 1-2i & 2+4i \\ 3-4i & 4+3i & 2-6i \\ 5 & 5+6i & 3 \end{bmatrix}$, *find* A^{θ}

Solution. We have $A = \begin{bmatrix} 2+3i & 1-2i & 2+4i \\ 3-4i & 4+3i & 2-6i \\ 5 & 5+6i & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \end{bmatrix}$

$$A^{\theta} = (\overline{A})' = \begin{bmatrix} 2-3i & 3+4i & 5 \\ 1+2i & 4-3i & 5-6i \\ 2-4i & 2+6i & 3 \end{bmatrix}$$ **Ans.**

EXERCISE 32.10

1. If the matrix $A = \begin{bmatrix} 1+i & 3-5i \\ 2i & 5 \end{bmatrix}$, find (*i*) $\overline{A}$ (*ii*) $(\overline{A})'$ (*iii*) A^{θ} (*iv*) $(A^{\theta})^{\theta}$

Show that

$$\overline{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is a unifary matrix, find A^{-1}

(*Vidyasagar University 2018*)

ANSWERS

1. (*i*) $\overline{A} = \begin{bmatrix} 1-i & 3+5i \\ -2i & 5 \end{bmatrix}$ (*ii*) $(\overline{A})' = \begin{bmatrix} 1-i & -2i \\ 3+5i & 5 \end{bmatrix}$

(*iii*) $A^\theta = \begin{bmatrix} 1-i & -2i \\ 3+5i & 5 \end{bmatrix}$ (*iv*) $(A^\theta)^\theta = \begin{bmatrix} 1+i & 3-5i \\ 2i & 5 \end{bmatrix}$

32.25 HERMITIAN MATRIX

Definition. A square matrix $A = [a_{ij}]$ is said to be Hermitian if the (i, j)th element of A, *i.e.*,

$$a_{ij} = \overline{a}_{ji} \quad \text{for all } i \text{ and } j.$$

For example $\begin{bmatrix} 2 & 3+4i \\ 3-4i & 1 \end{bmatrix}, \begin{bmatrix} a & b-id \\ b+id & c \end{bmatrix}$

Hence all the elements of the principal diagonal are real.

A necessary and sufficient condition for a matrix A to be Hermitian is that $A = A^\theta$.

Example 1. *Prove that the following*

(*i*) $(A^\theta)^\theta = A$ (*ii*) $(A + B)^\theta = A^\theta + B^\theta$ (*iii*) $(kA)^\theta = \overline{k}\, A^\theta$ (*iv*) $(AB)^\theta = B^\theta \cdot A^\theta$

where A^θ and B^θ be the transposed conjugates of A and B respectively, A and B being conformable to multiplication.

Solution.

(*i*) $(A^\theta)^\theta = [\overline{\{(\overline{A})'\}}]' = [\overline{\overline{A}}] = A$ as $\{(\overline{A})'\}' = \overline{A}$

(*ii*) $(A+B)^\theta = (\overline{A+B})' = (\overline{A}+\overline{B})'$

$$= (\overline{A})' + (\overline{B})' = A^\theta + B^\theta$$

(*iii*) $(kA)^\theta = (\overline{kA})' = (\overline{k}\,\overline{A})' = \overline{k}(\overline{A})' = \overline{k}\, A^\theta$

(*iv*) $(AB)^\theta = (\overline{AB})' = (\overline{A} \cdot \overline{B})' = (\overline{B})' \cdot (\overline{A})' = B^\theta \cdot A^\theta$ **Proved.**

Example 2. *Prove that matrix* $A = \begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix}$ *is Hermitian.*

Solution. $\overline{A} = \begin{bmatrix} 1 & 1+i & 2 \\ 1-i & 3 & -i \\ 2 & i & 0 \end{bmatrix} \Rightarrow (\overline{A})' = \begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix}$

$A^\theta = A$ $\Rightarrow A$ is Hermitian matrix. **Proved.**

Example 3. *Show that* $A = \begin{bmatrix} -i & 3+2i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -2i \end{bmatrix}$ *is Skew-Hermitian matrix.*

Solution. $\overline{A} = \begin{bmatrix} i & 3-2i & -2+i \\ -3-2i & 0 & 3+4i \\ 2+i & -3+4i & 2i \end{bmatrix}$

$$(\overline{A})' = \begin{bmatrix} i & -3-2i & 2+i \\ 3-2i & 0 & -3+4i \\ -2+i & 3+4i & 2i \end{bmatrix}$$

$$\Rightarrow \qquad A^\theta = \begin{bmatrix} i & -3-2i & 2+i \\ 3-2i & 0 & -3+4i \\ -2+i & 3+4i & 2i \end{bmatrix} \qquad [\because \; A^\theta = (\overline{A})']$$

$$= -\begin{bmatrix} -i & 3+2i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -2i \end{bmatrix} = -A$$

$A^\theta = -A$

$\Rightarrow$ A is Skew-Hermitian matrix. **Proved.**

Example 4. *Show that the matrix $B^\theta AB$ is Hermitian or Skew-Hermitian according as A is Hermitian or Skew-Hermitian.*

Solution. (i) Let A be Hermitian $\Rightarrow A^\theta = A$

Now
$$(B^\theta AB)^\theta = (AB)^\theta (B^\theta)^\theta$$
$$= B^\theta \cdot A^\theta \cdot B$$
$$= B^\theta \cdot A \cdot B \qquad (A^\theta = A)$$

Hence, $B^\theta AB$ is Hermitian.

(ii) Let A be Skew-Hermitian $\Rightarrow A^\theta = -A$

Now,
$$(B^\theta AB)^\theta = (AB)^\theta \cdot (B^\theta)^\theta$$
$$= B^\theta \cdot A^\theta \cdot B$$
$$= -B^\theta A \cdot B \qquad (A^\theta = -A)$$

Hence $B^\theta AB$ is Skew-Hermitian. **Proved.**

THE CHARACTERISTIC ROOTS OF A HERMITIAN MATRIX ARE ALL REAL

We know that matrix A is Hermitian if

$$A^\theta = A \text{ i.e., where } A^\theta \; (\overline{A}') \text{ or } (\overline{A})'$$

Also
$$(\lambda A)^\theta = \overline{\lambda} A^\theta \text{ and } (AB)^\theta = B^\theta A^\theta.$$

If λ is a characteristic root of matrix A then $AX = \lambda X$. …(1)

$\therefore \qquad (AX)^\theta = (\lambda X)^\theta \qquad$ or $\qquad X^\theta A^\theta \; \overline{\lambda} \, X^\theta.$

But A is Hermitian. $\quad \therefore A^\theta = A$.

$\therefore \quad X^\theta A = \overline{\lambda} X^\theta \quad \therefore \quad X^\theta AX = \overline{\lambda} X^\theta X$ ...(2)

Again from (1) $X^\theta AX = X^\theta \lambda X = \lambda X^\theta X$ ...(3)

Hence from (2) and (3) we conclude that $\overline{\lambda} = \lambda$ showing that λ is real.

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Deduction 1. *From above we conclude that characteristic roots of real symmetric matrix are all real, as in this case, real symmetric matrix will be Hermitian.*

For symmetric, we know that $A' = A$. $\left(\overline{A'}\right) = \overline{A}$

$\therefore$ or $A^\theta = A$ $\overline{A} = A$ as A is real. Rest as above.

32.26 SKEW-HERMITIAN MATRIX

Definition. A square matrix $A = (a_{ij})$ is said to be Skew-Hermitian matrix if the (i, j)th element of A is equal to the negative of the conjugate complex of the (j, i)th element of A, *i.e.*, $a_{ij} = -\overline{a}_{ji}$ for all i and j.

If A is a Skew-Hermitian matrix, then

$$a_{ii} = -\overline{a}_{ii}$$

$$a_{ii} + \overline{a}_{ii} = 0$$

Obviously, a_{ii} is either a pure imaginary number or must be zero.

For example $\begin{bmatrix} 0 & -3+4i \\ 3+4i & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & a-ib \\ -a-ib & 0 \end{bmatrix}$ are Skew-Hermitian matrixes.

A necessary and sufficient condition for a matrix A to be Skew-Hermitian is that $A^\theta = -A$.

Deduction 2. *Characteristic roots of a skew Hermitian matrix is either zero or a pure imaginary numbers.* **(*D.U. III Sem. 2012, April 2010*)**

If A is skew Hermitian, then iA is Hermitian.

Also λ be a characteristic root of A then $AX = \lambda X$.

$\therefore$ $(i\,.A)\,X = (i\lambda)\,X.$

Above shows that $i\lambda$ is characteristic root of matrix iA, which is Hermitian and hence $i\lambda$ should be real, which will be possible if λ is either pure imaginary or zero.

Example 1. *Show that every square matrix can be expressed as R + iS uniquely where R and S are Hermitian matrices*.

Solution. Let A be any square matrix. It can be rewritten as

$$A = \left\{\frac{1}{2}(A + A^\theta)\right\} + i\left\{\frac{1}{2i}(A - A^\theta)\right\} = R + iS$$

where $R = \frac{1}{2}(A + A^\theta),\ S = \frac{1}{2i}(A - A^\theta)$

Now we have to show that R and S are Hermitian matrices.

$$R^\theta = \frac{1}{2}(A + A^\theta)^\theta = \frac{1}{2}[A^\theta + (A^\theta)^\theta] = \frac{1}{2}(A^\theta + A) = \frac{1}{2}(A + A^\theta) = R$$

Thus R is Hermitian matrix.

Now, $S^\theta = \left[\frac{1}{2i}(A - A^\theta)\right]^\theta = -\frac{1}{2i}(A - A^\theta)^\theta$

$$= -\frac{1}{2i}[A^\theta - (A^\theta)^\theta] = \frac{-1}{2i}(A^\theta - A) = \frac{1}{2i}(A - A^\theta) = S$$

Thus S is a Hermitian matrix.

Hence $A = R + iS$, where R and S are Hermitian matrices.

Now, we have to show its **uniqueness**.

Let $A = P + iQ$ be another expression, where P and Q are Hermitian matrices, *i.e.*,

$$P^\theta = P, Q^\theta = Q$$

Then $$A^\theta = (P+iQ)^\theta = P^\theta + (iQ)^\theta = P^\theta - iQ^\theta = P - iQ$$

$$A = P + iQ \text{ and } A^\theta = P - iQ$$

$$P = \frac{1}{2}(A + A^\theta) = R \text{ and } Q = \frac{1}{2i}(A - A^\theta) = S$$

Hence $A = R + iS$ is the unique expression, where R and S are Hermitian matrices. **Proved.**

Example 2. *Express the matrix* $A \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$ *as a sum of Hermitian and Skew Hermitian matrix.* **(*U.P.I Sem Dec 2009*)**

Solution. Here, we have

$$A = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix} \quad \text{... (1)}$$

$$\overline{A} = \begin{bmatrix} -i & 2+3i & 4-5i \\ 6-i & 0 & 4+5i \\ i & 2+i & 2-i \end{bmatrix}$$

$$(\overline{A}) = \begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix}$$

$$A^\theta = \begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix} \quad \text{... (2)}$$

On adding (1) & (2), we get

$$A + A^\theta = \begin{bmatrix} 0 & 8-4i & 4+6i \\ 8+4i & 0 & 6-4i \\ 4-6i & 6+4i & 4 \end{bmatrix}$$

Let $$R = \frac{1}{2}\left[A + A^\theta\right] = \begin{bmatrix} 0 & 4-2i & 2+3i \\ 4+2i & 0 & 3-2i \\ 2-3i & 3+2i & 2 \end{bmatrix} \quad \text{... (3)}$$

On subtracting (2) from (1), we get

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$$A - A^{\theta} = \begin{bmatrix} 2i & -4-2i & 4+4i \\ 4-2i & 0 & 2-6i \\ -4+4i & -2-6i & 2i \end{bmatrix}$$

$$\frac{1}{2}\left(A - A^{\theta}\right) = \begin{bmatrix} i & -2-i & 2+2i \\ 2-i & 0 & 1-3i \\ -2+2i & -1-3i & i \end{bmatrix} \qquad \text{... (4)}$$

From (3) and (4), we have

$$A = \underset{\text{Hermitian matrix}}{\begin{bmatrix} 0 & 4-2i & 2+3i \\ 4+2i & 0 & 3-2i \\ 2-3i & 3+2i & 2 \end{bmatrix}} + \underset{\text{Skew-Hermitian matrix}}{\begin{bmatrix} i & -2-i & 2+2i \\ 2-i & 0 & 1-3i \\ -2+2i & -1-3i & i \end{bmatrix}}$$

Example 3. *Express the matrix* $A = \begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix}$ *as the sum of Hermitian matrix and Skew-Hermitian matrix.*

Solution. $A = \begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} 1-i & 2 & 5+5i \\ -2i & 2-i & 4-2i \\ -1-i & -4 & 7 \end{bmatrix}$ ...(1)

$$(\bar{A})' = \begin{bmatrix} 1-i & -2i & -1-i \\ 2 & 2-i & -4 \\ 5+5i & 4-2i & 7 \end{bmatrix} \Rightarrow A^{\theta} = \begin{bmatrix} 1-i & -2i & -1-i \\ 2 & 2-i & -4 \\ 5+5i & 4-2i & 7 \end{bmatrix} \qquad \text{...(2)}$$

On adding (1) and (2), we get

$$A + A^{\theta} = \begin{bmatrix} 2 & 2-2i & 4-6i \\ 2+2i & 4 & 2i \\ 4+6i & -2i & 14 \end{bmatrix}$$

Let $R = \frac{1}{2}(A + A^{\theta}) = \begin{bmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & i \\ 2+3i & -i & 7 \end{bmatrix}$ ...(3)

On subtracting (2) from (1), we get

$$A - A^{\theta} = \begin{bmatrix} 2i & 2+2i & 6-4i \\ -2+2i & 2i & 8+2i \\ -6-4i & -8+2i & 0 \end{bmatrix}$$

Let $S \frac{1}{2}(A - A^{\theta}) = \begin{bmatrix} i & 1+i & 3-2i \\ -1+i & i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix}$ ...(4)

From (3) and (4), we have

$$A = \begin{bmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & i \\ 2+3i & -i & 7 \end{bmatrix} + \begin{bmatrix} i & 1+i & 3-2i \\ -1+i & i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix}$$ **Ans.**

Hermitian matrix   Skew-Hermitian matrix

Example 4. *For any square matrix, $AA^{\theta} = I$ show that $A^{\theta} = I$.*

Solution. $AA^{\theta} = I$ So A is invertible. (given)

Let B be another matrix such that

$$AB = BA = I \quad \text{...(1)}$$

Now $B = BI = B(AA^{\theta})\ (AA^{\theta} = I)$

$= (BA)\ A^{\theta}$

$= IA^{\theta} = A^{\theta}$ [Using (1)]

We know that $BA = I$ [From (1)]

Putting the value of B from (2) in (1), we get

$\Rightarrow$ $A^{\theta}\ A = I$ **Proved.**

CHARACTERISTIC ROOTS OF A SKEW-HERMITIAN MATRIX IS EITHER ZERO OR PURELY AN IMAGINARY NUMBER

Since A is a skew-Hermitian matrix: $\therefore$ $i\,A$ is Hermitian matrix.

Let λ be a characteristic root of A.

Then, $AX = \lambda X \Rightarrow (iA)\ X = (i\lambda)\ X$

$\Rightarrow$ $i\lambda$ is a characteristic root of matrix iA.

But $i\lambda$ is a characteristic root of Hermitian matrix.

Therefore, $i\lambda$ should be real.

Hence, λ is either zero or purely imaginary. **Proved.**

32.27 PERIODIC MATRIX

A square matrix is said to be periodic, if $A^{k+1} = A$, where k is a positive integer. If k is the least positive integer for which $A^{k+1} = A$, then A is said to be of period k.

32.28 IDEMPOTENT MATRIX

A square matrix is said to be idempotent provided $A^2 = A$.

PROVE THAT THE EIGEN VALUES OF AN IDEMPOTENT MATRIX ARE EITHER ZERO OR UNITY

(***R.G.P.V. Bhopal I Semester June 2007***)

Solution. Let A be an idempotent matrix

$\therefore \qquad A^2 = A$

Let λ be a characteristic root of A and the corresponding vector be X. Hencce $X \neq 0$ and

$$AX = \lambda X \qquad ...(1)$$

$$\Rightarrow \quad A(AX) = A(\lambda X) = \lambda(AX)$$

$$\Rightarrow \quad (AA)X = \lambda(\lambda X) \qquad [\because \text{ From (1), } AX = \lambda X]$$

$$\Rightarrow \quad A^2X = \lambda^2 X$$

$$\Rightarrow \quad AX = \lambda^2 X \qquad [\because A^2 = A]$$

$$\Rightarrow \quad \lambda X = \lambda^2 X \qquad [\text{From (1) } AX = \lambda X]$$

$$\Rightarrow \quad (\lambda^2 - \lambda)X = 0 \quad \Rightarrow \quad \lambda^2 - \lambda = 0$$

$$\Rightarrow \quad \lambda(\lambda - 1) = 0 \quad \Rightarrow \quad \lambda^2 = 0, 1 \qquad [\because X \neq 0]$$

Hence, the eigen values of an idempotent matrix are either zero or unity. **Proved.**

Example. *Determine all the idempotent diagonal matrices of order n.*

Solution. Let A = diag. $[d_1, d_2, d_3, ... d_n]$ be an idempotent matrix of order n.

Here, for the matrix 'A' to be idempotent $A^2 = A$

$$\begin{bmatrix} d_1 & 0 & 0........0 \\ 0 & d_2 & 0........0 \\ 0 & 0 & d_3.......0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0........d_n \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0........0 \\ 0 & d_2 & 0........0 \\ 0 & 0 & d_3.......0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0........d_n \end{bmatrix} = \begin{bmatrix} d_1 & 0 & 0........0 \\ 0 & d_2 & 0........0 \\ 0 & 0 & d_3.......0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0........d_n \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} d_1^2 & 0 & 0........0 \\ 0 & d_2^2 & 0........0 \\ 0 & 0 & d_3^2.......0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0........d_n^2 \end{bmatrix} = \begin{bmatrix} d_1 & 0 & 0........0 \\ 0 & d_2 & 0........0 \\ 0 & 0 & d_3.......0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0........d_n \end{bmatrix} \quad \therefore$$

$$d_1^2 = d_1; \quad d_2^2 = d_2.........d_n^2 = d_n$$

i.e., $\qquad d_1 = 0, 1; \; d_2 = 0, 1; \; d_3 = 0, 1 \text{ .... } d_n = 0, 1.$

Hence diag. $[d_1, d_2, d_3 \ldots d_n]$, is the required idempotent matrix where

$$d_1 = d_2 = d_3 = ... \; d_n = 0 \text{ or } 1.$$ **Ans.**

EXERCISE 32.11

1. Which of the following matrices are Hermitian:

(a) $\begin{bmatrix} 1 & 2+i & 3-i \\ 2+i & 2 & 4-i \\ 3+i & 4+i & 3 \end{bmatrix}$ $\qquad (b)$ $\begin{bmatrix} 2i & 3 & 1 \\ 4 & -1 & 6 \\ 3 & 7 & 2i \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 2-i & 5+2i \\ 2+i & 1 & 2-5i \\ 5-2i & 2+5i & 2 \end{bmatrix}$ $\qquad (d)$ $\begin{bmatrix} 0 & i & 3 \\ -7 & 0 & 5i \\ 3i & 1 & 0 \end{bmatrix}$

2. Which of the following matrices are Skew-Hermitian:

(*a*) $\begin{bmatrix} 2i & -3 & 4 \\ 3 & 3i & -5 \\ -4 & 5 & 4i \end{bmatrix}$ (*b*) $\begin{bmatrix} 3i & -1 & 2 \\ 1 & 2i & -6 \\ 4 & 6 & -3i \end{bmatrix}$

(*c*) $\begin{bmatrix} 0 & 1-i & 2+3i \\ -1-i & 0 & 6i \\ -2+3i & 6i & 4i \end{bmatrix}$ (*d*) $\begin{bmatrix} 1 & 3 & 7+i \\ 3i & -i & 6 \\ 7-i & 8 & 0 \end{bmatrix}$

3. Give an example of a matrix which is Skew-symmetric but not Skew-Hermitian.

4. If A be a Hermitian matrix, show that iA is Skew-Hermitian. Also show that if B be a Skew-Hermitian matrix, then iB must be Hermitian.

5. If A and B are Hermitian matrices, then show that $AB + BA$ is Hermitian and $AB - BA$ is Skew-Hermitian.

6. If A is any square matrix, show that $A + A^{\theta}$ is Hermitian.

7. If $H = \begin{bmatrix} 3 & 5+2i & -3 \\ 5-2i & 7 & 4i \\ -3 & -4i & 5 \end{bmatrix}$, show that H is a Hermitian matrix.

Verify that iH is a Skew-Hermitian matrix.

8. Show that for any complex square matrix A,

(*i*) $(A + A^*)$ is a Hermitian matrix, where $A^* = A^{-T}$

(*ii*) $(A - A^*)$ is Skew-Hermitian matrix.

(*iii*) A^*A^* and A^*A are Hermitian matrices.

9. Show that any complex square matrix can be uniquely expressed as the sum of a Hermitian matrix and a Skew-Hermitian matrix.

10. Express $A = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$ as the sum of Hermitian and Skew-Hermitian matrices.

11. Prove that the latent roots of a Hermitian matrix are all real.

12. If $A = \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$ show that AA^* is a Hermitian matrix; where A^* is the conjugate transpose of A. **(*AMIETE June 2010*)**

ANSWERS

1. (*c*) **2.** (*a*), (*c*) **3.** $\begin{bmatrix} 0 & 2+3i \\ -2-3i & 0 \end{bmatrix}$

32.29 UNITARY MATRIX

A square matrix A is said to be unitary matrix if

$$A \cdot A^{\theta} = A^{\theta} A = I$$

Example 1. *If A is a unitary matrix, show that A^T is also unitary.*

Solution. $A \cdot A^\theta = A^\theta A = I$, since A is a unitary matrix.

$$(AA^\theta)^\theta = (A^\theta A)^\theta = I^\theta \qquad (I^\theta = I)$$

$$(AA^\theta)^\theta = (A^\theta A)^\theta = I$$

$$(A^\theta)^\theta A^\theta = A^\theta (A^\theta)^\theta = I$$

$$AA^\theta = A^\theta A = I \qquad [\text{since } (A^\theta)^\theta = A]$$

$$(AA^\theta)^T = (A^\theta A)^T = (I)^T$$

$$(A^\theta)^T A^T = A^T (A^\theta)^T = I$$

$$(A^T)^\theta \cdot A^T = A^T (A^T)^\theta = I$$

Hence, A^T is a unitary matrix. **Proved.**

Example 2. *If A is a unitary matrix, show that A^{-1} is also unitary.* (*DU, III Sem. 2012*)

Solution. $AA^\theta = A^\theta A = I$, since A is a unitary matrix.

$$(AA^\theta)^{-1} = (A^\theta \cdot A)^{-1} = (I)^{-1} \quad \text{taking inverse}$$

$$(A^\theta)^{-1} \cdot A^{-1} = A^{-1}(A^\theta)^{-1} = I$$

$$(A^{-1})^\theta \cdot A^{-1} = A^{-1}(A^{-1})^\theta = I$$

Hence, A^{-1} is a unitary matrix. **Proved.**

Example 3. *If A and B are two unitary matrices, show that AB is a unitary matrix.*

Solution. $A \cdot A^\theta = A^\theta A = I$ since A is a unitary matrix. ...(1)

Similarly, $B \cdot B^\theta = B^\theta B = I$ ...(2)

Now,
$$(AB)(AB)^\theta = (AB)(B^\theta \cdot A^\theta)$$
$$= A(BB^\theta) \cdot A^\theta$$
$$= A I A^\theta \qquad [\text{From (2)}]$$
$$= AA^\theta = I \qquad [\text{From (1)}]$$

Again,
$$(AB)^\theta \cdot (AB) = (B^\theta \cdot A^\theta)(AB)$$
$$= B^\theta (A^\theta A) B \qquad [\text{From (1)}]$$
$$= B^\theta I B$$
$$= B^\theta B$$
$$= I \qquad [\text{From (2)}]$$

Hence, AB is a unitary matrix. **Proved.**

Example 4. *Prove that the matrix* $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ *is unitary.*

Solution. Let $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$$A^\theta = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$A^\theta \cdot A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \times \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1+(1+1) & (1+i)-(1+i) \\ (1-i)-1(1-i) & (1+1)+1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence, A is a unitary matrix. **Proved.**

Example 5. *Show that the matrix* $A = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$ *is a unitary matrix if*

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$$ **(*U.P. I Semester Dec.* 2005)**

Solution. We have,

$$A = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$$

$$A^\theta = \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix}$$

We know that, a square matrix A is said to be unitary if $A\,A^\theta = I$

$$\begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}\begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \alpha^2 + \gamma^2 + \beta^2 + \delta^2 & \alpha\beta - i\alpha\delta + i\beta\gamma + \gamma\delta - \alpha\beta - i\beta\gamma + i\alpha\delta - \delta\gamma \\ \alpha\beta - i\beta\gamma + i\alpha\delta + \gamma\delta - \alpha\beta - i\alpha\delta + i\beta\gamma - \delta\gamma & \beta^2 + \delta^2 + \alpha^2 + \gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} \alpha^2 + \beta^2 + \gamma^2 + \delta^2 & 0 \\ 0 & \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$$ **Proved.**

Example 6. *Define a unitary matrix. If* $N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$ *is a matrix, then show that* $(I - N)(I + N)^{-1}$ *is a unitary matrix, where I is an identity matrix.*

(*D.U. April 2010*)

Solution. Unitary matrix: A square matrix 'A' is said to be unitary if $A^\theta A = I$, where $A^\theta = (\overline{A})^T$ and I is an identity matrix.

we have $$N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$$

$$I - N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \qquad ...(1)$$

Now we have to find $(I + N)^{-1}$

$$I + N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$|I + N| = 1 - (-1 - 4) = 6$$

$$\text{Adj. } (I+N) = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$(I+N)^{-1} = \frac{Adj(I+N)}{|I+N|} = \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \quad ...(2)$$

For unitary matrix $A^{\theta} A = I$

From (1) and (2), we get

$$\therefore (I-N)(I+N)^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} = B \text{ (say)}$$

Now $\quad (\overline{B})^T = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$

$$(\overline{B})^T B = \frac{1}{36}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} = I.$$

Hence the result. **Proved.**

35.30. THE MODULUS OF EACH CHARACTERISTIC ROOT OF A UNITARY MATRIX IS UNITY

(D.U. April 2010 U.P.)

Solution. Suppose A is a unitary matrix. Then

$$A^{\theta} A = I.$$

Let λ be a characteristic root of A. Then

$$AX = \lambda X \quad ...(1)$$

Taking conjugate transpose of both sides of (1), we get

$$(AX)^{\theta} = \overline{\lambda} X^{\theta} \quad ...(2)$$

$$\Rightarrow \quad X^{\theta} A^{\theta} = \overline{\lambda} X^{\theta}$$

From (1) and (2), we have

$$\Rightarrow \quad (X^{\theta} A^{\theta})(AX) = \overline{\lambda}\lambda X^{\theta} X$$

$$\Rightarrow \quad X^{\theta}(A^{\theta} A) X = \overline{\lambda}\lambda X^{\theta} X$$

$$\Rightarrow \quad X^{\theta} I X = \overline{\lambda}\lambda X^{\theta} X \quad (\because A^{\theta} . A = I)$$

$$\Rightarrow \quad X^{\theta} X = \overline{\lambda}\lambda X^{\theta} X$$

$$\Rightarrow \quad X^{\theta} X(\overline{\lambda}\lambda - 1) = 0 \quad ...(3)$$

Since $X^{\theta} X \neq 0$ therefore (3) gives

$\lambda\overline{\lambda} - 1 = 0.$ or $\lambda\overline{\lambda} = 1$ or $|\lambda|^2 = 1 \quad \Rightarrow \quad |\lambda| = 1$ **Proved.**

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EXERCISE 32.12

1. Show that the matrix $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ is unitary.

2. Prove that a real matrix is unitary if it is orthogonal.

3. Prove that the following matrix is unitary:

$$\begin{bmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) \\ \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \end{bmatrix}$$

4. Show that $U = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$ is a unitary matrix, where ω is the complex cube root of unity.

5. Prove that the latent roots of a unitary matrix have unit modulus.

6. Verify that the matrix

$$A = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

has eigen values with unit modulus.

Tick (✓) the correct answer:

7. If λ is an eigen value of the matrix 'M' then for the matrix $(M - \lambda I)$, which of the following statement (*s*) is/are coorrect ?

(*i*) Skew symmetric (*ii*) Non singular (*iii*) Singular (*iv*) None of these

(*U.P. I Sem. Dec. 2009*)

8. A square matrix A is idempotent if :

(*i*) $A' = A$ (*ii*) $A' = -A$ (*iii*) $A^2 = A$ (*iv*) $A^2 = I$

(*R.G.P.V. Bhopal I Semester June 2007*)

9. If a square matrix U such tha $\overline{U}' = U^{-1}$ then U is

(*i*) Orthogonal (*ii*) Unitary (*iii*) Symmetric (*iv*) Hermitian

(*R.G.P.V. Bhopal I Semester June 2007*)

10. If λ is an eigen value of a non-singular matrix A then the eigen value of A^{-1} is

(*i*) 1/λ (*ii*) λ (*iii*) –λ (*iv*) –1/λ

(*AMIETE June 2010*)

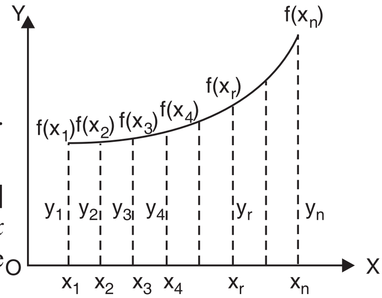
ANSWERS

7. (*iii*)	**8.** (*iii*)
9. (*ii*)	**10.** (*i*)

33 CHAPTER

Multiple Integrals

33.1 DOUBLE INTEGRATION



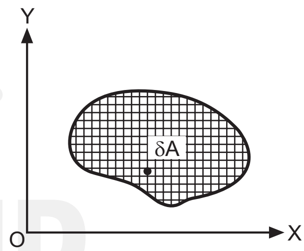
We know that

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \delta x\to 0}} [f(x_1)\,\delta x_1 + f(x_2)\,\delta x_2 + f(x_3)\,\delta x_3 + \ldots + f(x_n)\,\delta x_n]$$

Let us consider a function $f(x, y)$ of two variable x and y defined in the finite region A of xy-plane. Divide the region A into elementary areas.

$$\delta A_1,\ \delta A_2,\ \delta A_3,\ \ldots\ldots\ \delta A_n$$

Then $$\iint_A f(x,y)\,dA = \lim_{\substack{n\to\infty \\ \delta A\to 0}} \left[\begin{matrix} f(x_1,y_1)\,\delta A_1 + f(x_2,y_2) \\ \delta A_2 + \ldots.. + f(x_n,y_n)\,\delta A_n \end{matrix}\right]$$



33.2 EVALUATION OF DOUBLE INTEGRAL

Double integral over region A may be evaluated by two successive integrations.

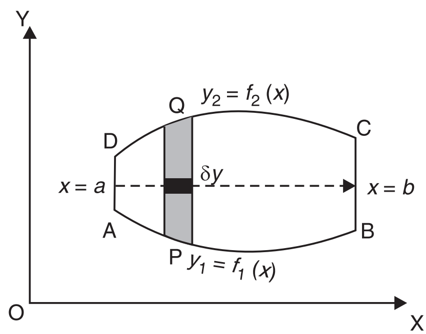
If A is described as $f_1(x) \le y \le f_2(x)$ $[y_1 \le y \le y_2]$

and $a \le x \le b$,

Then $$\iint_A f(x,y)\,dA = \int_a^b \int_{y_1}^{y_2} f(x,y)\,dx\,dy$$

(1) First Method

$$\iint_A f(x,y)\,dA = \int_a^b \boxed{\int_{y_1}^{y_2} f(x,y)\,dy}\,dx$$



$f(x, y)$ is first integrated with respect to y treating x as constant between the limits a and b.

In the region we take an elementary area $\delta x \delta y$. Then integration w.r.t y (x keeping constant), converts small rectangle $\delta x \delta y$ into a strip PQ $(y\ \delta x)$. While the integration of the result w.r.t. x corresponding to the sliding to the strip PQ, from AD to BC covering the whole region $ABCD$.

Second method

$$\iint_A f(x,y)\,dxdy = \int_c^d \boxed{\int_{x_1}^{x_2} f(x,y)\,dx}\,dy$$

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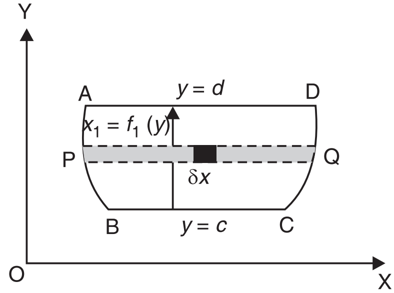
Here $f(x,y)$ is first integrated w.r.t. x keeping y constant between the limits x_1 and x_2 and then the resulting expression is integrated with respect to y between the limits c and d.



Take a small area $\delta x \delta y$. The integration w.r.t. x between the limits x_1, x_2 keeping y fixed indicates that integration is done, along PQ. Then the integration of result w.r.t y corresponds to sliding the strips PQ from BC to AD covering the whole region $ABCD$.

Note. For constant limits, it does not matter whether we first integrate w.r.t. x and then w.r.t. y or vice versa.

Example 1. *Evaluate* $\int_0^1 \int_0^x (x^2 + y^2)\, dA$, *where dA indicates small area in xy-plane.*

(*Gujarat I Semester Jan. 2009*)

Solution. Let

$$I = \int_0^1 \int_0^x (x^2 + y^2)\, dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^x dx$$

$$= \int_0^1 \left[x^2 (x - 0) + \frac{1}{3}(x^3 - 0) \right] dx = \int_0^1 \left[x^3 + \frac{x^3}{3} \right] dx$$

$$= \int_0^1 \frac{4}{3} x^3\, dx = \frac{4}{3} \left[\frac{x^4}{4} \right]_0^1 = \frac{1}{3}[1 - 0] = \frac{1}{3} \text{ sq units.}$$ **Ans.**

Example 2. *Evaluate* $\int_{-1}^{1} \int_0^{1-x} x^{1/3}\, y^{-1/2}\, (1 - x - y)^{1/2}\, dy\, dx$.

Solution. Here, we have

$$I = \int_{-1}^{1} \int_0^{1-x} x^{1/3}\, y^{-1/2}\, (1 - x - y)^{1/2}\, dy\, dx \qquad ...(1)$$

Putting $(1 - x) = c$ in (1), we get

$$I = \int_{-1}^{1} x^{1/3}\, dx \int_0^c y^{-1/2}\, (c - y)^{1/2}\, dy \qquad ...(2)$$

Again putting $y = ct \Rightarrow dy = c\, dt$ in (2), we get

$$I = \int_{-1}^{1} x^{\frac{1}{3}}\, dx \int_0^1 c^{-\frac{1}{2}}\, t^{-\frac{1}{2}}\, (c - ct)^{\frac{1}{2}}\, c\, dt$$

$$= \int_{-1}^{1} x^{1/3}\, dx \int_0^1 c^{-1/2}\, t^{-1/2}\, c^{1/2}\, (1 - t)^{1/2}\, c\, dt$$

$$= \int_{-1}^{1} c\, x^{1/3}\, dx \int_0^1 t^{-1/2} (1 - t)^{1/2}\, dt = \int_{-1}^{1} c\, x^{1/3}\, dx \int_0^1 t^{1/2 - 1}\, (1 - t)^{3/2 - 1}\, dt$$

$$= \int_{-1}^{1} c\, x^{\frac{1}{3}}\, dx\, \beta\left(\frac{1}{2}, \frac{3}{2}\right) \quad \left[\int_0^1 x^{l-1}\, (1 - x)^{m-1}\, dx = \beta(l, m) \right]$$

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$$= \int_{-1}^{1} c\, x^{1/3}\, dx \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{3}{2}\right)} = \int_{-1}^{1} c\, x^{1/3}\, dx \frac{\Gamma\left(\frac{1}{2}\right) \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = \int_{-1}^{1} c\, x^{1/3}\, dx \frac{\sqrt{\pi}\, \frac{1}{2}\sqrt{\pi}}{1}$$

$$= \int_{-1}^{1} c\, x^{1/3} \frac{\pi}{2}\, dx = \frac{\pi}{2}\int_{-1}^{1} x^{1/3} \,.\, c\, dx$$

Putting the value of c, we get

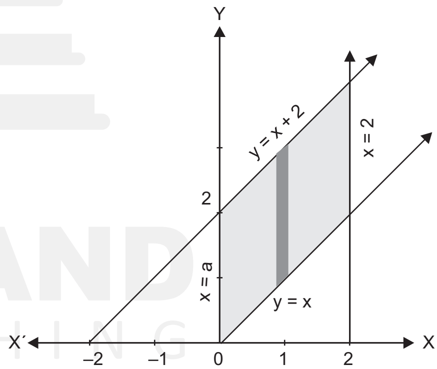
$$I = \frac{\pi}{2}\int_{-1}^{1} x^{1/3}(1-x)\, dx = \frac{\pi}{2}\int_{-1}^{1} (x^{1/3} - x^{4/3})\, dx = \frac{\pi}{2}\left[\frac{x^{4/3}}{\frac{4}{3}} - \frac{x^{7/3}}{\frac{7}{3}}\right]_{-1}^{1}$$

$$= \frac{\pi}{2}\left[\frac{3}{4}(1) - \frac{3}{7}(1) - \frac{3}{4}(-1) + \frac{3}{7}(-1)\right] = \frac{\pi}{2}\left[\frac{9}{14}\right] = \frac{9\pi}{28} \qquad \textbf{Ans.}$$

Example 3. *Evaluate* $= \iint_R (x+y)\, dy\, dx$ *is the region bounded by* $x = 0$, $x = 2$, $y = x$, $y = x + 2$. **(*Gujarat I Semester Jan. 2009*)**

Solution. Let $I = \iint_R (x+y)\, dy\, dx$

The limits are $x = 0$, $x = 2$, $y = x$ and $y = x + 2$



$$I = \int_0^2 dx \int_x^{x+2} (x+y)\, dy = \int_0^2 \left[xy + \frac{y^2}{2}\right]_x^{x+2} dx$$

$$= \int_0^2 \left[x(x+2) + \frac{1}{2}(x+2)^2 - x^2 - \frac{x^2}{2}\right] dx$$

$$= \int_0^2 \left[x^2 + 2x + \frac{1}{2}(x^2+4x+4) - x^2 - \frac{x^2}{2}\right] dx$$

$$= \int_0^2 [2x + 2x + 2]\, dx$$

$$= 2\int_0^2 (2x+1)\, dx = 2[x^2 + x]_0^2 = 2\,[4 + 2] = 12 \qquad \textbf{Ans.}$$

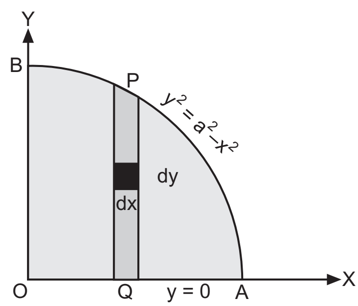
Example 4. *Evaluate* $\iint_R xy\, dx\, dy$ *where R is the quadrant of the circle* $x^2 + y^2 = a^2$ *where* $x \geq 0$ *and* $y \geq 0$.

Solution. Let the region of integration be the first quadrant of the circle OAB.

$$\iint_R xy\, dx\, dy \quad (x^2 + y^2 = a^2 \ \Rightarrow\ y = \sqrt{a^2 - x^2})$$

First we integrate w.r.t. y and then w.r.t. x.

The limits for y are 0 and $\sqrt{a^2 - x^2}$ and for x, 0 to a.



$$= \int_0^a x\, dx \int_0^{\sqrt{a^2-x^2}} y\, dy = \int_0^a x\, dx \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}}$$

$$= \frac{1}{2}\int_0^a x(a^2 - x^2)\,dx \quad = \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{a^4}{8}$$ **Ans.**

Example 5. *Evaluate* $\iint_s \sqrt{xy - y^2}\, dy\, dx$, *where S is a triangle with vertices (0, 0), (10, 1) and (1, 1).*

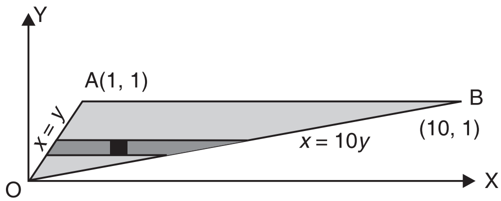
Solution. Let the verties of a triangle *OBA* be (0, 0) (10, 1) and (1, 1).

Equation of *OA* is $x = y$.

Equation of *OB* is $x = 10\,y$.

The region of ΔOBA, given by the limits

$y \le x \le 10\,y$ and $0 \le y \le 1$.



$$\iint_s \sqrt{xy - y^2}\, dy\, dx = \int_0^1 dy \int_y^{10y} (xy - y^2)^{1/2}\, dx$$

$$= \int_0^1 dy \left[\frac{2}{3}\frac{1}{y}(xy - y^2)^{3/2}\right]_y^{10y} = \int_0^1 \frac{2}{3}\frac{1}{y}(9y^2)^{3/2}\, dy = 18\int_0^1 y^2\, dy$$

$$= 18\left[\frac{y^3}{3}\right]_0^1 = \frac{18}{3} = 6$$ **Ans.**

Example 6. *Evaluate* $\iint_A x^2\, dx\, dy$, *where A is the region in the first quadrant bounded by the hyperbola xy = 16 and the lines y = x, y = 0 and x = 8.*

Solution. The line *OP*, $y = x$ and the curve *PS*, $xy = 16$ intersect at (4, 4).

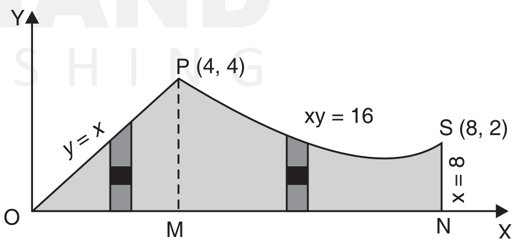
The line *SN*, $x = 8$ intersects the hyperbola at *S* (8, 2). $y = 0$ is x-axis.

The area *A* is shown shaded.

Divide the area in to two part by *PM* perpendicular to *OX*.

For the area OMP, y varies from 0 to x, and then x varies from 0 to 4.

For the area PMNS, y -series from 0 to $16/x$ and then x varies from 4 to 8.



$$\therefore \qquad \iint_A x^2\, dx\, dy = \int_0^4\int_0^x x^2\, dx\, dy + \int_4^8\int_0^{16/x} x^2\, dx\, dxy$$

$$= \int_0^4 x^2\, dx \int_0^x dy + \int_4^8 x^2\, dx \int_0^{16/x} dy = \int_0^4 x^2\,[y]_0^x\, dx + \int_4^8 x^2\,[y]_0^{16/x}\, dx$$

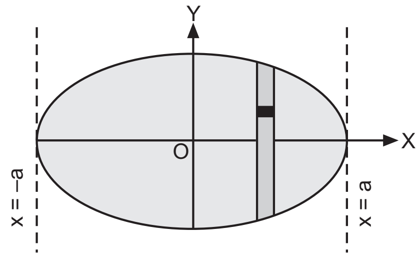
$$= \int_0^4 x^3\, dx + \int_4^8 16\,x\, dx = \left[\frac{x^4}{4}\right]_0^4 + 16\left[\frac{x^2}{2}\right]_4^8 = 64 + 8\,(8^2 - 4^2) = 64 + 384 = 448.$$ **Ans.**

Example 7. *Evaluate* $\iint (x + y)^2\, dx\, dy$ *over the area bounded by the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(*U.P. Ist Semester Compartment 2004*)

Solution. For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \quad \frac{y}{b} = \pm\sqrt{1-\frac{x^2}{a^2}} \quad \Rightarrow \quad y = \pm\frac{b}{a}\sqrt{a^2-x^2}$$



$\therefore$ The region of integration can be expressed as

$-a \le x \le a$ and $-\frac{b}{a}\sqrt{a^2-x^2} \le y \le \frac{b}{a}\sqrt{a^2-x^2}$

$$\therefore \quad \iint (x+y)^2\, dx\, dy = \iint (x^2+y^2+2xy)\, dx\, dy$$

$$= \int_{-a}^{a}\int_{(-b/a)\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x^2+y^2+2xy)\, dx\, dy$$

$$= \int_{-a}^{a}\int_{(-b/a)\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x^2+y^2)\, dx\, dy \int_{-a}^{a}\int_{(-b/a)\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} 2xy\, dy\, dx$$

$$= \int_{-a}^{a}\int_{0}^{b/a\sqrt{a^2-x^2}} 2\,(x^2+y^2)\, dy\, dx + 0$$

[Since (x^2+y^2) is an even function of y and $2xy$ is an odd function of y]

$$= \int_{-a}^{a}\left[2\left(x^2y+\frac{y^3}{3}\right)\right]_0^{\left(\frac{b}{a}\right)\sqrt{a^2-x^2}} dx = 2\int_{-a}^{a}\left[x^2\times\frac{b}{a}\sqrt{a^2-x^2}+\frac{1}{3}\frac{b^3}{a^3}(a^2-x^2)^{3/2}\right]dx$$

$$= 4\int_0^a\left[\frac{b}{a}x^2\sqrt{a^2-x^2}+\frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right]dx$$

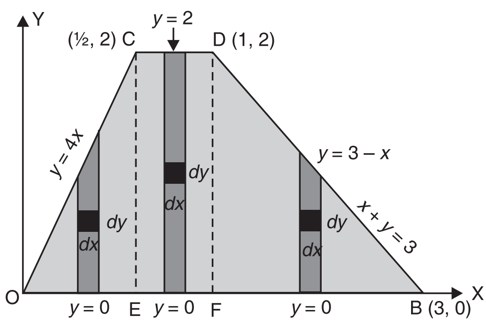
[On putting $x = a\sin\theta$ and $dx = a\cos\theta\, d\theta$]

$$= 4\int_0^{\frac{\pi}{2}}\left(\frac{b}{a}.a^2\sin^2\theta.a\cos\theta+\frac{b^3}{3a^3}a^3\cos^3\theta\right)\times a\cos\theta\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}}\left(a^3b\sin^2\theta\cos^2\theta+\frac{ab^3}{3}\cos^4\theta\right)d\theta = 4\left[a^3b.\frac{1}{4}.\frac{1}{2}.\frac{\pi}{2}+\frac{ab^3}{3}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}\right]$$

$$= \frac{\pi}{4}(a^3b+ab^3) = \frac{\pi}{4}ab\,(a^2+b^2)$$ **Ans.**

Example 8. *Evaluate* $\iint (x^2+y^2)\, dx\, dy$ *throughout the area enclosed by n the curves $y = 4x$, $x + y = 3$, $y = 0$ and $y = 2$.*



Solution. Let OC represent $y = 4x$; BD, $x + y = 3$; OB, $y = 0$, and CD, $y = 2$. The given integral is to be evaluated over the area A of the trapezium $OCDB$.
Area $OCDB$ consists of area OCE, area $ECDF$ and area FDB.

The co-ordinates of C, D and B are $\left(\frac{1}{2}, 2\right)$ (1, 2) and (3, 0) respectively.

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$$\therefore \quad \iint_A (x^2 + y^2)\, dy\, dx$$

$$= \iint_{OCE} (x^2 + y^2)\, dy\, dx + \iint_{ECDE} (x^2 + y^2)\, dy\, dx + \iint_{FDB} (x^2 + y^2)\, dy\, dx$$

$$= \underbrace{\int_0^{1/2} dx \int_0^{4x} (x^2 + y^2)\, dy}_{I_1} + \underbrace{\int_{1/2}^1 dx \int_0^2 (x^2 + y^2)\, dy}_{I_2} + \underbrace{\int_1^3 dx \int_0^{3-x} (x^2 + y^2)\, dy}_{I_3}$$

Now, $I_1 = \int_0^{1/2} dx \int_0^{4x} (x^2 + y^2)\, dy = \int_0^{1/2} \left[x^2 y + \frac{y^3}{3} \right]_0^{4x} dx = \int_0^{1/2} \frac{76}{3} x^3\, dx$

$$= \frac{76}{3} \int_0^{1/2} x^3\, dx = \frac{76}{3} \left[\frac{x^4}{4} \right]_0^{1/2} = \frac{76}{3} \left[\frac{1}{4} \cdot \frac{1}{16} \right] = \frac{19}{48}$$

$$I_2 = \int_{1/2}^1 dx \int_0^2 (x^2 + y^2)\, dy = \int_{1/2}^1 \left[x^2 y + \frac{y^3}{3} \right]_0^2 dx = \int_{1/2}^1 \left(2x^2 + \frac{8}{3} \right) dx$$

$$= \left[\frac{2x^3}{3} + \frac{8}{3} x \right]_{1/2}^1 = \left[\left(\frac{2}{3} + \frac{8}{3} \right) - \left(\frac{2}{3} \cdot \frac{1}{8} + \frac{8}{3} \cdot \frac{1}{2} \right) \right] = \frac{23}{12}$$

$$I_3 = \int_1^3 dx \int_0^{3-x} (x^2 + y^2)\, dy = \int_1^3 \left[x^2 y + \frac{y^3}{3} \right]_0^{3-x} dx = \int_1^3 \left[x^2 (3 - x) + \frac{(3-x)^3}{3} \right] dx$$

$$= \int_1^3 \left[3x^2 - x^3 + \frac{(3-x)^3}{3} \right] dx = \left[x^3 - \frac{x^4}{4} - \frac{(3-x)^4}{3 \times 4} \right]_1^3 = \left[27 - \frac{81}{4} - 0 - 1 + \frac{1}{4} + \frac{16}{12} \right] = \frac{22}{3}$$

$$\therefore \quad \int_A \int (x^2 + y^2)\, dy\, dx = I_1 + I_2 + I_3 = \frac{19}{48} + \frac{23}{12} + \frac{22}{3} = \frac{463}{48} = 9\frac{31}{48}.$$ **Ans.**

EXERCISE 33.1

Evaluate

1. $\int_0^2 \int_0^{x^2} e^{\frac{y}{x}}\, dy\, dx$

2. $\int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy$

3. $\int_0^a \int_0^{\sqrt{a^2 - y^2}} dx\, dy$

4. $\int_0^1 \int_{y^2}^{y} (1 + xy^2)\, dx\, dy$

5. $\int_0^{2a} \int_0^{\sqrt{2ax - x}} xy\, dy\, dx$

6. $\int_0^{2a} \int_0^{\sqrt{2ax - x^2}} x^2\, dy\, dx$

7. $\int_0^a \int_0^{\sqrt{a^2 - x^2}} \sqrt{a^2 - x^2 - y^2}\, dy\, dx$

8. $\int_0^1 \int_0^{\sqrt{\frac{1}{2}(1 - y^2)}} \frac{dx\, dy}{\sqrt{1 - x^2 - y^2}}$

9. $\int_0^a \int_0^{\sqrt{a^2 - x^2}} \frac{dx\, dy}{(1 + e^y)\sqrt{a^2 - x^2 - y^2}}$

10. $\int_0^a \int_y^a \frac{x\, dx\, dy}{\sqrt{x^2 + y^2}}$

11. $\int_{x=0}^{1} \int_{y=0}^{2} (x^2 + 3xy^2)\, dx\, dy$ **(*A.M.I.E.T.E. June 2009*)**

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12. $\iint_A (5 - 2x - y)\, dx\, dy,$ where A is given by $y = 0$, $x + 2y = 3$, $x = y^2$.

13. $\iint_A xy\, dx\, dy,$ where A is given by $x^2 + y^2 - 2x = 0$, $y^2 = 2x$, $y = x$.

14. $\iint_A \sqrt{4x^2 - y^2}\, dx\, dy,$ where A is the triangle given by $y = 0$, $y = x$ and $x = 1$.

15. $\iint_R x^2\, dx\, dy,$ where R is the two-dimensional region bounded by the curves $y = x$ and $y = x^2$.

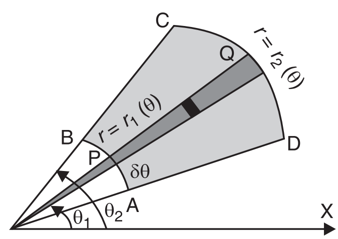
16. $\iint_A \sqrt{xy(1 + x - y)}\, dx\, dy$ where A is the area bounded by $x = 0$, $y = 0$ and $x + y = 1$.

ANSWERS

1. $e^2 - 1$	**2.** $\frac{a^4}{6}$	**3.** $\frac{\pi a^2}{4}$
4. $\frac{41}{210}$	**5.** $\frac{2a^4}{3}$	**6.** $\frac{5\pi a^4}{8}$
7. $\frac{\pi a^3}{4}$	**8.** $\frac{\pi}{4}$	**9.** $\frac{\pi}{2} \log \frac{2e^a}{1 + e^a}$
10. $\frac{a^2}{2} \log (\sqrt{2} + 1)$	**11** $\frac{14}{3}$	**12.** $\frac{217}{60}$
13. $\frac{7}{12}$	**14.** $\frac{1}{3}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$	**15.** $\frac{1}{20}$
16. $\frac{2\pi}{105}$		

33.3 EVALUATION OF DOUBLE INTEGRALS IN POLAR CO-ORDINATES

We have to evalaute $\int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta)\, dr\, d\theta$ over the region bounded by the straight lines



$\theta = \theta_1$ and $\theta = \theta_2$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$. We first integrate with respect to r between the limits $r = r_1(\theta)$ and $r = r_2(\theta)$ and taking θ as constant. Then the resulting expression is integrated with respect to θ between the limits $\theta = \theta_1$ and $\theta = \theta_2$.

The area of integration is *ABCD*. On integrating first with respect to r, the strip extends from P to Q and the integration with respect to θ means the rotation of this strip PQ from AD to BC.

Example 9. *Transform the integral to cartesian form and hence evaluate*

$$\int_0^{\pi} \int_0^{a} r^3 \sin\theta \cos\theta\, dr\, d\theta.$$

Solution. Here, we have

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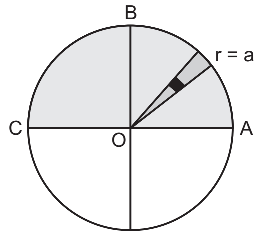
$$\int_0^{\pi}\int_0^{a} r^3 \sin\theta\cos\theta\, dr\, d\theta \qquad ...(1)$$



Here the region *i.e.*, semicircle *ABC* of integration is bounded by $r = 0$, *i.e.*, x-axis.

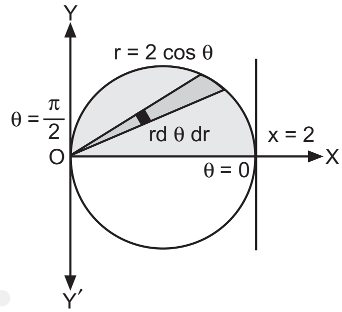
$r = a$ *i.e.*, circle, $\theta = 0$ and $\theta = \pi$ *i.e.*, x-axis in the second quadrant.

$$\int\int (r\sin\theta)(r\cos\theta)(r\, d\theta\, dr)$$

Putting $x = r\cos\theta$, $y = r\sin\theta$, $dx\, dy = r\, d\theta\, dr$ in (1), we get

$$\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} xy\, dy\, dx = \int_{-a}^{a} x\, dx \int_0^{\sqrt{a^2-x^2}} y\, dy$$

$$= \int_{-a}^{a} x\, dx \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} = \int_{-a}^{a} x\, dx\, \frac{(a^2-x^2)}{2}$$



$$= \frac{1}{2}\int_{-a}^{a} (a^2 x - x^3)\, dx = 0 \textbf{ Ans.} \quad \left[\begin{array}{l}\text{Since } f(x) \text{ is odd function} \\ \int_{-a}^{a} f(x)\, dx = 0\end{array}\right]$$

Example 10. *Evaluate* $\int_0^2\int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$

Solution. $\int_0^2\int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$

Limits of $\qquad y = \sqrt{2x-x^2} \qquad \Rightarrow y^2 = 2x - x^2$

$\Rightarrow\ x^2 + y^2 - 2x = 0 \qquad ...(1)$

(1) represents a circle whose centre is (1, 0) and radius = 1.

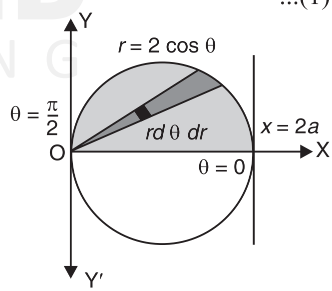
Lower limit of y is 0 *i.e.,* x-axis.

Region of integration is upper half circle.

Let us convert (1) into polar co-ordinate by putting

$$x = r\cos\theta,\ y = r\sin\theta$$

$$r^2 - 2r\cos\theta = 0 \Rightarrow r = 2\cos\theta$$



Limits of r are 0 to $2\cos\theta$

Limits of θ are 0 to $\frac{\pi}{2}$

$$\int_0^2\int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx = \int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^2\,(r\, d\theta\, dr) = \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} r^3\, dr = \int_0^{\frac{\pi}{2}} d\theta \left[\frac{r^4}{4}\right]_0^{2\cos\theta}$$

$$= 4\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta = 4 \times \frac{3\times 1\times\pi}{4\times 2\times 2} = \frac{3\pi}{4} \qquad \textbf{Ans.}$$

Example 11. *Evaluate* $\int_0^2\int_0^{\sqrt{2x-x^2}} \frac{x\, dy\, dx}{\sqrt{x^2+y^2}}$ *by changing to polar coordinates.*

(***A.M.I.E.T.E. 2017***)

Solution. In the given integral, y varies from 0 to $\sqrt{2x-x^2}$ and x varies from 0 to 2.

$$y = \sqrt{2x - x^2}$$

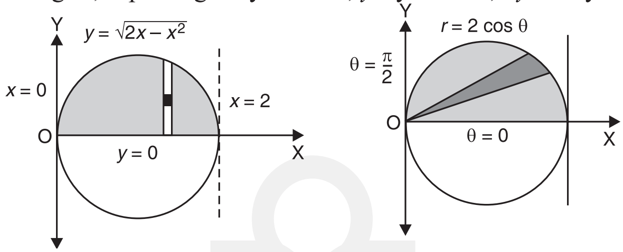
$\Rightarrow$ $y^2 = 2x - x^2$

$\Rightarrow$ $x^2 + y^2 = 2x$

In polar co-ordinates, we have $r^2 = 2r \cos\theta \quad \Rightarrow \quad r = 2\cos\theta.$

$\therefore$ For the region of integarion, r varies from 0 to $2\cos\theta$ and θ varies from 0 to $\frac{\pi}{2}$.

In the given integral, replacing x by $r\cos\theta$, y by $r\sin\theta$, $dy\,dx$ by $r\,dr\,d\theta$, we have



$$I = \int_0^{\pi/2}\int_0^{2\cos\theta} \frac{r\cos\theta.\, r\,dr\,d\theta}{r} = \int_0^{\pi/2}\int_0^{2\cos\theta} r\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2} \cos\theta\left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta = \int_0^{\pi/2} 2\cos^3\theta\,d\theta = 2.\frac{2}{3} = \frac{4}{3}.$$ **Ans.**

EXERCISE 33.2

Evaluate the following:

1. $\int_0^{\pi}\int_0^{a(1-\cos\theta)} 2\pi r^2 \sin\theta\, d\theta\, dr$

2. $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2 \cos\theta\, dr\, d\theta$

3. $\iint_A \frac{r\,dr\,d\theta}{\sqrt{r^2+a^2}}$ where A is a loop of $r^2 = a^2\cos 2\theta$

4. $\iint_A r^2 \sin\theta\, d\theta\, dr$ where A is $r = 2a\cos\theta$ above initial line.

5. Calculate the integral $\iint \frac{(x-y)^2}{x^2+y^2}\,dx\,dy$ over the circle $x^2 + y^2 \le 1$.

6. $\iint (x^2 + y^2)\, x\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$ by changing to polar coordinates.

7. $\iint_R \sqrt{x^2+y^2}\, dx\, dy$ by changing to polar coordinates, R is the region in the xy-plane bounded by the circles $x^2 + y^2 = 4$ **(*AMIETE Dec. 2009*)**

8. Convert into polar coordinates

$$\int_0^{2a}\int_0^{2ax-x^2} dx\,dy$$

9. $\iint r^3\, dr\, d\theta$, over the area bounded between the circles $r = 2b\cos\theta$ and $r = 2b\cos\theta$.

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10. $\iint r \sin\theta \, dr \, d\theta$ over the area of the cardiod $r = a(1 + \cos\theta)$ above the initial line.

11. $\int\int_A x^2 \, dr \, d\theta$, where A is the area between the circles $r = a\cos\theta$ and $r = 2a\cos\theta$.

12. Transform the integral $\int_0^1 \int_0^x f(x, y)\, dy\, dx$ to the integral in polar co-ordinates.

ANSWERS

1. $\frac{8}{3}\pi a^3$	**2.** $\frac{5}{8}\pi a^3$	**3.** $2a - \frac{\pi a}{2}$	**4.** $\frac{2a^3}{3}$
5. $\pi - 2$	**6.** $\frac{a^2}{5}$	**7.** $\frac{38\pi}{3}$	
8. $\int_0^{\pi/2} \int_0^{2a\cos\theta} r\, d\theta\, dr$	**9.** $\frac{3\pi}{2}(a^4 - b^4)$	**10.** $\frac{5}{8}\pi a^3$	
11. $\frac{28a^3}{9}$	**12.** $\int_0^{\pi/4} \int_0^{\sec\theta} f(r, \theta)\, r\, d\theta\, dr$		

33.4 CHANGE OF ORDER OF INTEGRATION

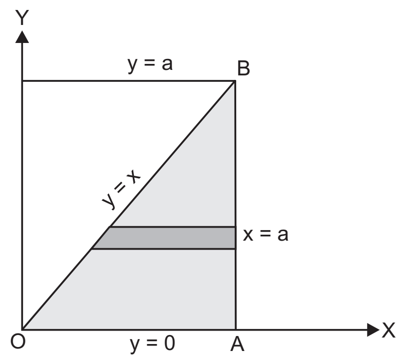
On changing the order of integration, the limits of integration change. To find the new limits, we draw the rough sketch of the region of integration.
Some of the problems connected with double integrals, which seem to be complicated, can be made easy to handle by a change in the order of integration.

Example 12. *Evaluate* $\int_0^a \int_y^a \frac{x}{x^2 + y^2}\, dx\, dy$ *by changing the order of integration.*

(*AMIETE June 2010, Nagpur University Summer 2008*)

Solution. Here we have

$$I = \int_0^a \int_y^a \frac{x}{x^2 + y^2}\, dx\, dy$$



Here $\quad x = a,\ x = y,\ y = 0$ and $y = a$

The area of integration is OAB.

On changing the order of integration Lower limit of $y = 0$ and

upper limit is $y = x$.

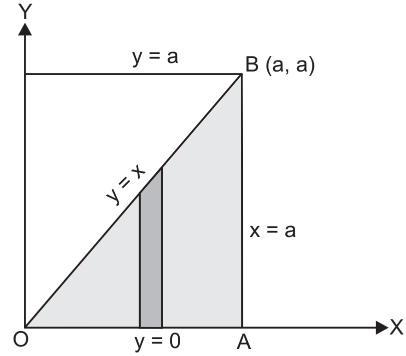
Lower limit of $x = 0$ and upper limit is $x = a$.

$$I = \int_0^a x\,dx \int_0^{y=x} \frac{1}{x^2 + y^2}\, dy$$

$$= \int_0^a x\,dx \left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^{y=x}$$

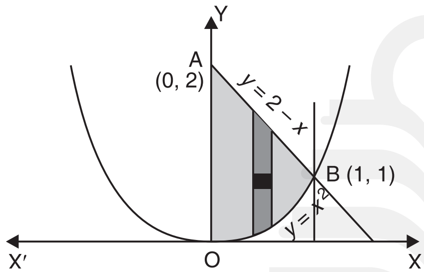
$$= \int_0^a \frac{x}{x} dx \left(\tan^{-1} \frac{x}{x} - \tan^{-1} 0 \right)$$

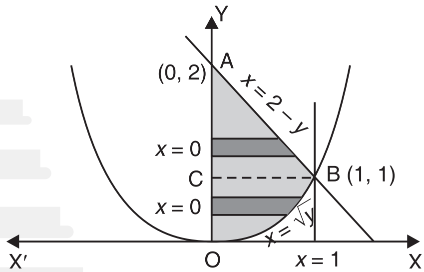
$$= \int_0^a dx \left(\frac{\pi}{4} \right) \quad = \frac{\pi}{4} [x]_0^a = \frac{a\pi}{4} \quad \textbf{Ans.}$$



Example 13. *Change the order of integration in* $I = \int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy$ *and hence evaluate the same.* **(*A.M.I.E.T.E. June 2010, 2009*)**

Solution. $I = \int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy$





The region of integration is shown by shaded portion in the figure bounded by parabola $y = x^2$ and the line $y = 2 - x$.

The point of intersection of the parabola $y = x^2$ and the line $y = 2 - x$ is B (1, 1).

In the figure below (left) we have taken a strip parallel to y-axis and the order of integration is

$$\int_0^1 x\, dx \int_{x^2}^{2-x} y\, dy$$

In the second figure above we have taken a strip parallel to x-axis in the area OBC and second strip in the area ABC. The limits of x in the area OBC are 0 and $\sqrt{y}$ and the limits of x in the area ABC are 0 and $2 - y$.

$$= \int_0^1 y\, dy \int_0^{\sqrt{y}} x\, dx + \int_1^2 y\, dx \int_0^{2-y} x\, dx = \int_0^1 y\, dy \left[\frac{x^2}{2} \right]_0^{\sqrt{y}} + \int_1^2 y\, dy \left[\frac{x^2}{2} \right]_0^{2-y}$$

$$= \frac{1}{2} \int_0^1 y^2\, dy + \frac{1}{2} \int_1^2 y\,(2-y)^2\, dy = \frac{1}{2} \left[\frac{y^3}{3} \right]_0^1 + \frac{1}{2} \int_1^2 (4y - 4y^2 + y^3)\, dy$$

$$= \frac{1}{6} + \frac{1}{2} \left[2y^2 - \frac{4}{3} y^3 + \frac{y^4}{4} \right]_1^2 = \frac{1}{6} + \frac{1}{2} \left[8 - \frac{32}{3} + 4 - 2 + \frac{4}{3} - \frac{1}{4} \right]$$

$$= \frac{1}{6} + \frac{1}{2} \left[\frac{96 - 128 + 48 - 24 + 16 - 3}{12} \right] = \frac{1}{6} + \frac{5}{24} = \frac{9}{24} = \frac{3}{8} \qquad \textbf{Ans.}$$

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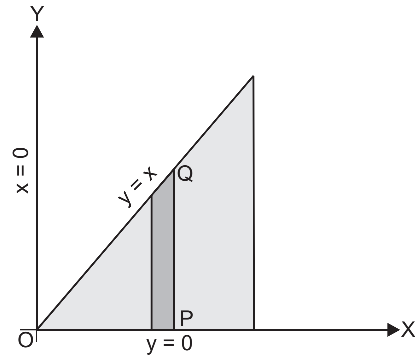
Example 14. *Evaluate the integral* $\int_0^\infty \int_0^x x \exp\left(-\frac{x^2}{y}\right) dx\, dy$ *by changing the order of integration* (*A.M.I.E.T.E., June 2017*)

Solution. Limits are given

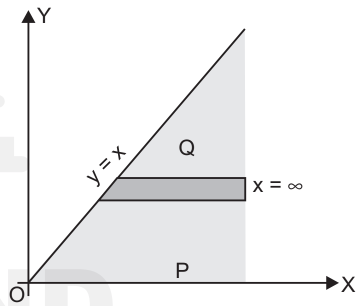
$$y = 0 \text{ and } y = x$$
$$x = 0 \text{ and } x = \infty$$



Here, the elementary strip *PQ* extends from $y = 0$ to $y = x$ and this vertical strip slides from $x = 0$ to $x = \infty$.

The region of integration is shown by shaded portion in the figure bounded by $y = 0$, $y = x$, $x = 0$ and $x = \infty$.

On changing the order of integration, we first integrate with respect to x along a horizontal strip *RS* which extends from $x = y$ to $x = \infty$ and this horizontal strip slides from $y = 0$ to $y = \infty$ to cover the given region of integration.



New limits :

$$x = y \quad \text{and} \quad x = \infty$$
$$y = 0 \quad \text{and} \quad y = \infty$$

We first integrate with respect to x.

Thus,

$$\int_0^\infty dy \int_y^\infty x e^{-\frac{x^2}{y}} dx = \int_0^\infty dy \int_y^\infty -\frac{y}{2}\left(-\frac{2x}{y} e^{-\frac{x^2}{y}}\right) dx$$

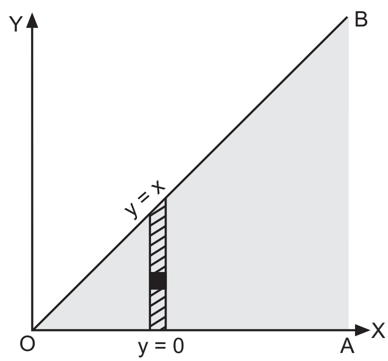
$$= \int_0^\infty dy \left[-\frac{y}{2} e^{-\frac{x^2}{y}}\right]_y^\infty = \int_0^\infty dy \left[0 + \frac{y}{2} e^{-\frac{y^2}{y}}\right] = \int_0^\infty \frac{y}{2} e^{-y} dy$$

$$= \left[\frac{y}{2}(-e^{-y}) - \left(\frac{1}{2}\right)(e^{-y})\right]_0^\infty \qquad \text{(Integrating by parts)}$$

$$= \left[(0-0) - \left(0 - \frac{1}{2}\right)\right] = \frac{1}{2} \qquad \textbf{Ans.}$$

Example 15. *Change the order of the integration*

$$\int_0^\infty \int_0^x e^{-xy}\, y\, dy\, dx \qquad (\textit{B.P.U.T.; I Semester 2008})$$



Solution. Here, we have

$$\int_0^\infty \int_0^x e^{-xy}\, y\, dy\, dx$$

Here the region *OAB* of integration is bounded by $y = 0$ (x-axis), $y = x$ (a straight line), $x = 0$, *i.e.*, y axis. A strip is drawn parallel to y-axis, y varies 0 to x and x varies 0 to ∞.

On changing the order of integration, first we integrate w.r.t. x and then w.r.t. y.

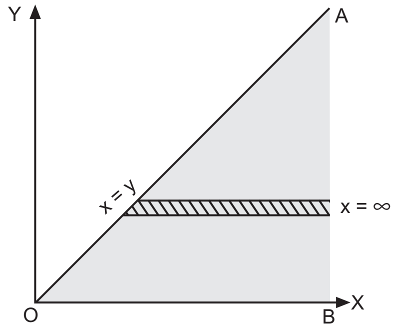
A strip is drawn parallel to x-axis. On this strip x varies from y to ∞ and y varies from 0 to ∞.

Hence $\int_0^\infty \int_0^x e^{-xy}\, y\, dy\, dx = \int_0^\infty y\, dy \int_y^\infty e^{-xy} dx$

$$= \int_0^\infty y\, dy \left(\frac{e^{-xy}}{-y}\right)_y^\infty$$

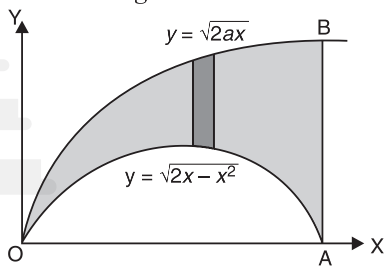
$$= \int_0^\infty \frac{y\, dy}{-y}[0 - e^{-y^2}]$$

$$= \int_0^\infty e^{-y^2}\, dy = \frac{1}{2}\sqrt{\pi} \quad \textbf{Ans.}$$



Example 16. *Change the order of integration in the double integral*

$$\int_0^{2a} \int_{\sqrt{2ax - x^2}}^{\sqrt{2ax}} V\, dx\, dy$$



Solution. Limits are given as

$$x = 0,\ x = 2a$$

$$y = \sqrt{2\,ax}$$

and $\quad y = \sqrt{2\,ax - x^2} \quad \Rightarrow \quad y^2 = 2ax - x^2$

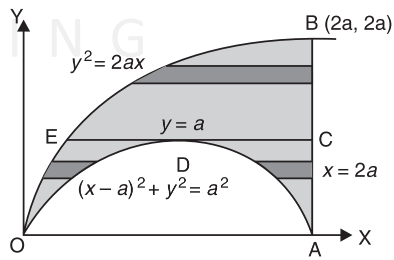
and $\quad (x - a)^2 + y^2 = a^2$

The area of integration is the shaded portion OAB. On changing the order of integration first we have to integrate w.r.t. x, The area of integration has three portions BCE, ODE and ACD.

$$\int_0^{2a} dx \int_{\sqrt{2\,ax - x^2}}^{\sqrt{2ax}} V\, dy$$

$$= \int_a^{2a} dy \int_{y^2/2a}^{2a} V\, dx + \int_0^a dy \int_{y^2/2a}^{a+\sqrt{a^2 - y^2}} V\, dx$$

$$+ \int_0^a dy \int_{a+\sqrt{a^2 - y^2}}^{2a} V\, dx \qquad \textbf{Ans.}$$



EXERCISE 33.3

Change the order of integration and hence evaluate the following:

1. $\int_0^a \int_0^x \frac{\cos y\, dy}{\sqrt{(a-x)(a-y)}}\, dx$

2. $\int_0^{2a} \int_{\frac{x^2}{4a}}^{3a - x} (x^2 + y^2)\, dy\, dx$

3. $\int_0^1 \int_{x^2}^{x} (x^2 + y^2)^{-1/2}\, dy\, dx$

4. $\int_0^a \int_{\sqrt{a^2 - y^2}}^{y + a} f(x, y)\, dx\, dy$

5. $\int_{-a}^{a} \int_0^{\sqrt{a^2 - y^2}} f(x, y)\, dx\, dy$

6. $\int_0^1 \int_x^{2-x} \frac{x}{y}\, dy\, dx$

7. $\int_0^b \int_y^a \frac{x\,dy\,dx}{x^2+y^2}$ (*M.P. 2003*) **8.** $\int_0^a \int_0^{bx/a} x\,dy\,dx$

9. $\int_0^5 \int_{2-x}^{2+x} f(x, y)\,dx\,dy$

10. $\int_0^\infty \int_{-y}^{y} (y^2 - x^2)\,e^{-y}\,dx\,dy$

11. $\int_{y=0}^{1} \int_{x=\sqrt{y}}^{2-y} \times yx\,dy$ (***A.M.I.E.T.E. June 2009***)

12. $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dx\,dy$ (***U.P. I Semester Dec. 2007***)

13. $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy\,dx\,dy$ [**Hint:** Put $x = a\sin^2\theta \Rightarrow dx = 2a\sin\theta\cos\theta\,d\theta$]

14. $\int_0^1 \int_{-1}^{1-y} x^{1/3}\,y^{-1/2}\,(1-x-y)^{1/2}\,dx\,dy$

15. $\int_0^{2a} dx \int_0^{\frac{x^2}{4a}} (x+y)^3\,dy$

16. $\int_0^1 \int_0^y (x^2+y^2)\,dx\,dy + \int_1^2 \int_0^{2-y} (x^2+y^2)\,dx\,dy$

17. $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\,dx\,dy$ by changing into polar coordinates.

(***U.P. I Semester Dec. 2007***)

18. $\int_0^1\int_1^2 \frac{1}{x^2+y^2}\,dx\,dy + \int_0^2\int_y^2 \frac{1}{x^2+y^2}\,dx\,dy = \int_R\int \frac{1}{x^2+y^2}\,dy\,dx$

Recognise the region R of integration on the R.H.S. and then evaluate the integral on the right in the order indicated.

19. Express as single integral and evaluate :

$$\int_0^{\frac{a}{\sqrt{2}}} \int_0^x x\,dx\,dy + \int_{\frac{a}{\sqrt{2}}}^{a} \int_0^{\sqrt{a^2-x^2}} x\,dx\,dy$$

20. Express as single integral and evaluate :

$$\int_0^1 \int_0^y (x^2+y^2)\,dx\,dy + \int_1^2 \int_0^{2-y} (x^2+y^2)\,dx\,dy$$

21. If $f(x, y)\,dx\,dy$, where R is the circle $x^2 + y^2 = a^2$, is R equivalent to the repeated integral.

(***AMIE winter 2001***)

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ANSWERS

1. (*a*) $\int_0^a dy \int_y^a \frac{\cos y\, dx}{\sqrt{(a-x)(a-y)}}$ (*b*) $2 \sin a$.

2. (*a*) $\int_0^a dy \int_0^{2\sqrt{ay}} (x^2+y^2)\, dx + \int_a^{3a} dy \int_0^{3a-y} (x^2+y^2)\, dx$ (*b*) $\frac{314\, a^4}{35}$.

3. $\int_0^1 dy \int_y^{\sqrt{y}} (x^2+y^2)^{-1/2}\, dx$.

4. $\int_0^a dx \int_{\sqrt{a^2-x^2}}^a f(x, y)\, dy + \int_a^{2a} dx \int_{x-a}^a f(x, y)\, dy$

5. $\int_0^a dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x, y)\, dy$

6. $\int_0^a \frac{dy}{y} \int_0^y x dx + \int_1^2 \frac{dy}{y} \int_0^{2-y} x dx; \log \frac{4}{e}$

8. (*a*) $\int_0^b dy \int_{ay/b}^a x\, dx$ (*b*) $\frac{1}{3} a^2 b$

9. $\int_0^2 dy \int_{2-y}^5 f(x, y)\, dx + \int_2^7 dy \int_{y-2}^5 f(x, y)\, dx$

10. $\int_{-\infty}^{\infty} dx \int_{-x}^x (y^2 - x^2)\, e^{-y}\, dy$

12. $\int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy + \int_0^{2a-y} xy\, dx\, dy, \frac{3a^2}{8}$

13. $\int_0^{2a} x\, dx \int_0^{\sqrt{a^2-(x-a)^2}} y\, dy, \frac{2}{3} a^4$

14. $\int_{-1}^1 x^{\frac{1}{3}}\, dx \int_0^{1-x} y^{-\frac{1}{2}} (1-x-y)^{\frac{1}{2}}\, dy, -\frac{3\pi}{7}$

15. $\int_0^a dy \int_{\sqrt{4ay}}^{2a} (x+y)^3 dx$

16. $\int_0^1 dx \int_x^{2-x} (x^2+y^2)\, dy, \frac{5}{3}$

17. $\frac{\pi a^5}{20}$

18. Region R is $x = 0, x = y, y = 1$ and $y = 2, \frac{\pi}{4} \log 2$.

19. $\int_0^{\frac{a}{\sqrt{2}}} dy \int_y^{\sqrt{a^2-y^2}} x\, dx, \frac{5a^3}{6\sqrt{2}}$

20. $\int_0^1 dx \int_x^{2-x} (x^2+y^2)\, dy, \frac{5}{3}$

21. $\int_0^{2\pi} \int_0^a f(r, \theta)\, r\, dr\, d\theta$.

33.5 CHANGE OF VARIABLES

Sometimes the problems of double integration can be solved easily by change of independent variables. Let the double integral as be $\iint_R f(x, y)\, dx\, dy$. It is to be changed by the new variables u, v.

The relation of x, y with u, v are given as $x = \phi(u, v)$, $y = \Psi(u, v)$. Then the double integration is converted into.

$\iint f\{\phi(u, v), \Psi(u, v)\}\,|J|\,du\,dv$, where

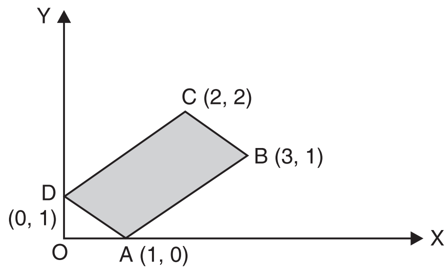
$$dx\,dy = |J|\,du\,dv = \frac{\partial(x, y)}{\partial(u, v)}\,du\,dv = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv$$

Example 17. *Evaluate* $\iint_R (x + y)^2\, dx\, dy$, *where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1), using the transformation* $u = x + y$ *and* $v = x - 2y$.

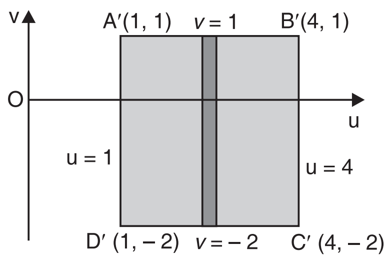
Solution. The region of integration is a parallelogram $ABCD$, where A (1, 0), B (3, 1), C (2, 2) and D (0, 1) in xy-plane.

The new region of integration is a rectangle $A'B'C'D'$ in uv-plane

xy-plane	$A \equiv (x, y)$ $A \equiv (1, 0)$	$B \equiv (x, y)$ $B \equiv (3, 1)$	$C \equiv (x, y)$ $C \equiv (2, 2)$	$D \equiv (x, y)$ $D \equiv (0, 1)$
uv-plane	$A' \equiv (u, v)$ $A' \equiv (x + y, x - 2y)$ $A' \equiv (1 + 0, 1 - 2 \times 0)$ $A' \equiv (1, 1)$	$B' \equiv (u, v)$ $B' \equiv (x + y, x - 2y)$ $B' \equiv (3 + 1, 3 - 2 \times 1)$ $B' \equiv (4, 1)$	$C' \equiv (u, v)$ $C' \equiv (x + y, x - 2y)$ $C' \equiv (2 + 2, 2 - 2 \times 2)$ $C' \equiv (4, -2)$	$D' \equiv (u, v)$ $D' \equiv (x + y, x - 2y)$ $D' \equiv (0 + 1, 0 - 2 \times 1)$ $D' \equiv (1, -2)$



(xy-plane)



(uv-plane)

and $\left\{\begin{array}{l} u = x + y \\ v = x - 2y \end{array}\right\} \Rightarrow x = \frac{1}{3}(2u + v)$ and $y = \frac{1}{3}(u - v)$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{3}$$

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$$dx\,dy = |J|\,du\,dv = \frac{1}{3}\,du\,dv$$

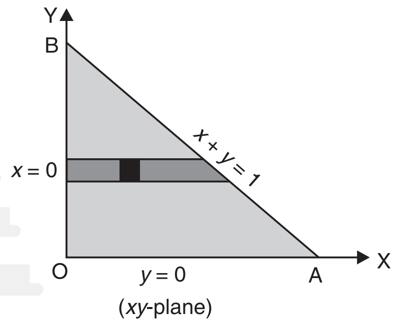
$$\iint_R (x+y)^2\,dx\,dy = \int_{-2}^{1}\int_{1}^{4} u^2 . \frac{1}{3}\,du\,dv = \int_{-2}^{1}\frac{1}{3}\left[\frac{u^3}{3}\right]_1^4 dv = \int_{-2}^{1} 7\,dv = 7\,[v]_2^1 = 7\times 3 = 21$$

Ans.

Example 18. *Using the transformation x + y = u, y = uv, show that*

$\iint [xy\,(1-x-y)]^{1/2}\,dx\,dy = \frac{2\pi}{105}$, integration being taken over the area of the tringle bounded by the lines x = 0, y = 0, x + y = 1.

Solution. $\iint [xy\,(1-x-y)]^{1/2}\,dx\,dy$



$x + y = u$ or $x = u - y = u - uv$,

$$dx\,dy = \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv = \begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} du\,dv$$

$$dx\,dy = \begin{vmatrix}1-v & -u\\ v & u\end{vmatrix} du\,dv = u\,du\,dv.$$

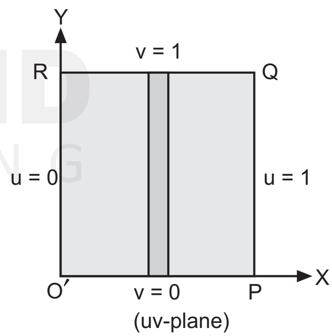
$x = 0 \Rightarrow u\,(1-v) = 0$

$\Rightarrow u = 0,\ v = 1$

$y = 0 \Rightarrow uv = 0$

$\Rightarrow u = 0,\ v = 0$

$x + y = u \Rightarrow u = 1$



Hence, the limits of u are from 0 to 1 and the limits of v are from 0 to 1.

The area of integration is a square *OPQR* in *uv*-plane.

On putting $x = u - uv$, $y = uv$, $dx\,dy = u\,du\,dv$ in (1), we get

$$\iint (u-uv)^{1/2}\,(uv)^{1/2}\,(1-u)^{1/2}\,u\,du\,dv$$

$$= \int_0^1 u^2\,(1-u)^{1/2}\,du \int_0^1 v^{1/2}\,(1-v)^{1/2}\,dv \int 4^{3-1}(1-4)^{\frac{3}{2}-1}_{du} \cdot V^{\frac{3}{2}-1}(1-b)^{\frac{3}{2}-1}_{dv} = \frac{\left|\overline{3}\times\right|\overline{\frac{3}{2}}}{\left|\overline{\frac{9}{2}}\right.} \times \frac{\left|\overline{\frac{3}{2}}\right|\overline{\frac{3}{2}}}{\left|\overline{3}\right.}$$

$$= \frac{2.\left|\overline{\frac{3}{2}}\right.}{\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\left|\overline{\frac{3}{2}}\right.} \times \frac{\frac{1}{2}\left|\overline{\frac{1}{2}}\right..\frac{1}{2}\left|\overline{\frac{1}{2}}\right.}{2} = \frac{2}{\frac{7}{2}.\frac{5}{2}.\frac{3}{2}} \times \frac{\frac{1}{2}\sqrt{\pi}.\frac{1}{2}\sqrt{\pi}}{2} = \frac{2\pi}{105}$$

Ans.

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EXERCISE 33.4

1. Evaluate $\int_0^\infty \int_0^\infty e^{-(x+y)} \sin\left(\frac{\pi y}{x+y}\right) dx\, dy$ by means of the transformation $u = x + y$, $v = y$ from (x, y) to (u, v)

2. Using the transformation $x + y = u$, $y = uv$, show that $\int_0^1 \int_0^{1-x} \frac{y}{e^{x+y}} dy\, dx = \frac{1}{2}(e-1)$

3. Using the transformation $u = x - y$, $v = x + y$, prove that $\iint_R \cos\frac{x-y}{x+y} dx\, dy = \frac{1}{2}\sin 1$ where R is bounded by $x = 0$, $y = 0$, $x + y = 1$

$$\left[\textbf{Hint: } x = \frac{1}{2}(u+v),\ y = \frac{1}{2}(v-u) \text{ so that } |J| = \frac{1}{2}\right]$$

ANSWERS

1. $\frac{1}{\pi}$

33.6 AREA IN CARTESIAN CO-ORDINATES

Let the curves AB and CD be $y_1 = f_1(x)$ and $y_2 = f_2(x)$.

Let the ordinates AD and BC be $x = a$ and $x = b$.

So the area enclosed by the two curves $y_1 = f_1(x)$ and $y_2 = f_2(x)$ and $x = a$ and $x = b$ is $ABCD$.

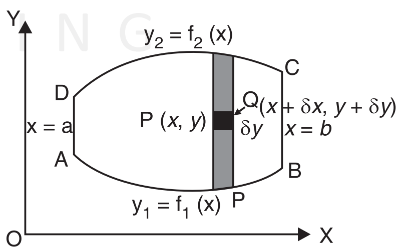
Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be two neighbouring points, then the area of the small rectangle $PQ = \delta x.\ \delta y$.

Area of the vertical strip $= \lim_{\delta y \to 0} \sum_{y_1}^{y_2} \delta x\, \delta y = \int_{y_1}^{y_2} dy$

δx the width of the strip is constant throughout.

If we add all the strips from $x = a$ to $x = b$, we get

The area $ABCD = \lim_{\delta x \to 0} \sum_a^b \delta x \int_{y_1}^{y_2} dy = \int_a^b dx \int_{y_1}^{y_2} dy$



$$\boxed{\textbf{Area} = \int_a^b \int_{y_1}^{y_2} dx\, dy}$$

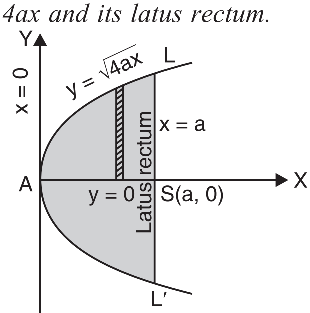
Example 19. *Find the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.*

Solution. Required area $= 2$ [(area (ASL)]

$$= 2\int_0^a \int_0^{2\sqrt{ax}} dy\, dx$$

$$= 2\int_0^a 2\sqrt{ax}\, dx$$

$$= 4\sqrt{a}\left(\frac{x^{3/2}}{3/2}\right)_0^a = \frac{8a^2}{3}$$



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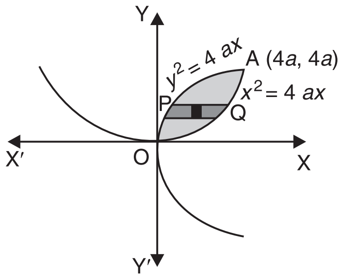
Example 20. *Find the area between the parabolas $y^2 = 4\,ax$ and $x^2 = 4\,ay$.*

Solution. $y^2 = 4ax$ ...(1)

$x^2 = 4ay$ ...(2)

On solving the equations (1) and (2) we get the point of intersection $(4a, 4a)$.

Divide the area into horizontal strips of width δy, x varies from $P, \dfrac{y^2}{4a}$ to $Q, \sqrt{4\,ay}$ and then y varies from $O(y = 0)$ to $A\ (y = 4a)$.



$$\therefore \text{ The required area} = \int_0^{4a} dy \int_{y^2/4a}^{\sqrt{4ay}} dx$$

$$= \int_0^{4a} dy\,[x]_{y^2/4a}^{\sqrt{4ay}} = \int_0^{4a} dy\left[\sqrt{4ay} - \frac{y^2}{4a}\right]$$

$$= \left[\sqrt{4a}\,\frac{y^{3/2}}{\frac{3}{2}} - \frac{y^3}{12\,a}\right]_0^{4a}$$

$$= \left[\frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{(4a)^3}{12a}\right] = \left[\frac{32}{3}a^2 - \frac{16}{3}a^2\right] = \frac{16}{3}a^2 \qquad \textbf{Ans.}$$

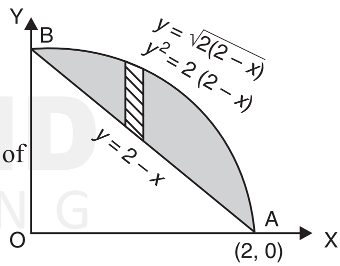
Example 21. *Find by double integration the area enclosed by the pair of curves*

$$y = 2 - x \textit{ and } y^2 = 2(2 - x)$$

Solution. $y = 2 - x$

$y^2 = 2\,(2 - x)$

On solving the equations (1) and (2), we get the points of intersection (2, 0) and (0, 2).



$$A = \int\int dx\,dy$$

$$\text{The required area} = \int_0^2 dx \int_{2-x}^{\sqrt{2(2-x)}} dy = \int_0^2 dx\,[y]_{2-x}^{\sqrt{2(2-x)}} = \int_0^2 dx\,[\sqrt{4 - 2x} - 2 + x]$$

$$= \left[\frac{2}{3 \times -2}(4 - 2x)^{3/2} - 2x + \frac{x^2}{2}\right]_0^2$$

$$= \left[-\frac{1}{3}(4 - 2x)^{3/2} - 2x + \frac{x^2}{2}\right]_0^2 = \left(-4 + \frac{4}{2}\right) + \frac{8}{3} = \frac{2}{3} \qquad \textbf{Ans.}$$

EXERCISE 33.5

Use double integration in the following questions:

1. Find the area bounded by $y = x - 2$ and $y^2 = 2x + 4$.
2. Find the area between the circle $x^2 + y^2 = a^2$ and the line $x + y = a$ in the first quadrant.

3. Find the area of a plate in the form of quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

4. Find the area included between the curves $y^2 = 4a\,(x + a)$ and $y^2 = 4b(b - x)$. (*A.M.I.E., Summer 2001*)

5. Find the area bounded by (*a*) $y^2 = 4 - x$ and $y^2 = x$.
(*b*) $x - 2y + 4 = 0$, $x + y - 5 = 0$, $y = 0$ **(*A.M.I.E., Winter 2001*)**

6. Find the area enclosed by the leminscate $r^2 = a^2 \cos 2\,\theta$.

7. Find the area common to the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = 2ax$.

8. Find the area included between the curves $y = x^2 - 6x + 3$ and $y = 2x + 9$.
(*A.M.I.E., Summer 2001*)

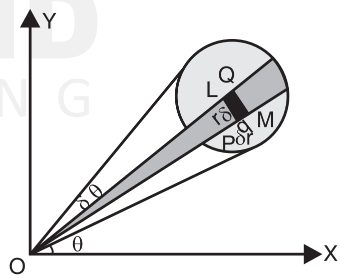
9. Determine the area of region bounded by the curves $xy = 2$, $4y = x^2$, $y = 4$.

ANSWERS

1. 18.	**2.** $(\pi - 2)a^2/4$	**3.** $\frac{\pi\, ab}{4}$	**4.** $\frac{\sqrt[8]{ab}}{3}$
5. (*a*) $\frac{16\sqrt{2}}{3}$	(*b*) $\frac{27}{2}$	**6.** a^2	**7.** $\left[\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right]a^2$
8. $\frac{88\sqrt{22}}{3}$	**9.** $\frac{28}{3} - 4\log 2$		

33.7 AREA IN POLAR CO-ORDINATES

$$\text{Area} = \iint r\, d\theta\, dr$$

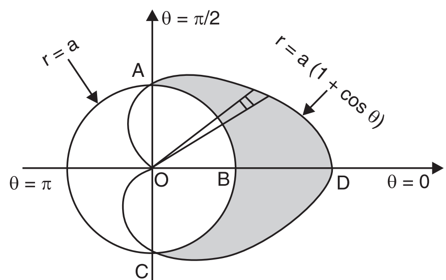


Let us consider the area enclosed by the curve $r = f(\theta)$.
Let $P\,(r, \theta)$, $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring points.
Draw ares *PL* and *QM*, radii *r* and $r + \delta r$.

$$PL = r\delta\theta,\ PM = \delta r$$

Area of rectangle $PLQM = PL \times PM = (r\delta\theta)\,(\delta r) = r\,\delta\theta\,\delta r$.
The whole arca *A* is composed of such small rectangles.
Hence,

$$A = \lim_{\substack{\partial r \to 0 \\ \delta\theta \to 0}} \sum \sum r\,\delta\theta.\,\delta r = \iint r\, d\theta\, dr$$



Example 22. *Find by double integration, the area lying inside the cardioid $r = a\,(1 + \cos\theta)$ and outside the circle $r = a$.*

Solution. $r = a\,(1 + \cos\theta)$ ...(1)
$r = a$ ...(2)

Solving (1) and (2), by eliminating *r*, we get

$$a(1 + \cos\theta) = a \quad\Rightarrow\quad 1 + \cos\theta = 1$$

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$$\cos\theta = 0 \quad \Rightarrow \quad \theta = -\frac{\pi}{2} \text{ or } \frac{\pi}{2}$$

limits of r are a and $a(1 + \cos\theta)$

limits of θ are $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

Required area = Area ABCDA

$$= \int_{-\pi/2}^{\pi/2}\int_{r \text{ for circle}}^{a(1+\cos\theta) \text{ for cardioid}} r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_{a}^{a(1+\cos\theta)} r dr d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left(\frac{r^2}{2}\right)_a^{a(1+\cos\theta)} d\theta = \frac{a^2}{2}\int_{-\pi/2}^{\pi/2}[(1+\cos\theta)^2 - 1]\,d\theta$$

$$= \frac{a^2}{2}\int_{-\pi/2}^{\pi/2}(\cos^2\theta + 2\cos\theta)\,d\theta = a^2\int_0^{\pi/2}(\cos^2\theta + 2\cos\theta)\,d\theta$$

$$= a^2\int_0^{\pi/2}(\cos^2\theta + 2\cos\theta)\,d\theta = a^2\left[\int_0^{\pi/2}\cos^2\theta\,d\theta + 2\int_0^{\pi/2}\cos\theta\,d\theta\right]$$

$$= a^2\left[\frac{\pi}{4} + 2\,(\sin\theta)_0^{\pi/2}\right] = a^2\left[\frac{\pi}{4} + 2\right] = \frac{a^2}{4}(\pi + 8)$$ **Ans.**

Example 23. *Find by double integration, the area lying inside the circle r = a sin θ and outside the cardioid r = a (1 – cos θ).*

Solution. We have,

$$r = a\sin\theta \quad \ldots(1)$$
$$r = a\,(1 - \cos\theta) \quad \ldots(2)$$

Solving (1) and (2) by eliminating r, we have

$$\sin\theta = 1 - \cos\theta \Rightarrow \sin\theta + \cos\theta = 1$$

Squaring above, we get

$$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 1$$

$$\Rightarrow \quad 1 + \sin 2\theta = 1 \Rightarrow \sin 2\theta = 0 \Rightarrow 2\theta = 0 \text{ or } \pi \Rightarrow \theta = 0 \text{ or } \frac{\pi}{2}$$

The required area is shaded portion in the fig.

Limits of r are $a(1 - \cos\theta)$ and $a\sin\theta$, limits of θ are 0 and $\frac{\pi}{2}$.

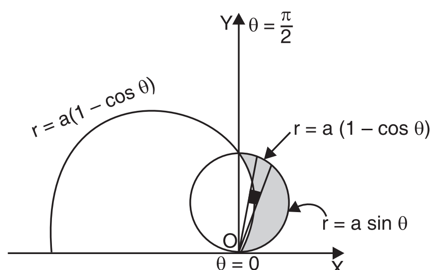
$$\text{Required area} = \int_0^{\frac{\pi}{2}}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta = \frac{1}{2}\int_0^{\pi/2} a^2\,[\sin^2\theta - (1-\cos\theta)^2]\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}(\sin^2\theta - 1 - \cos^2\theta + 2\cos\theta)\,d\theta$$

$$= \frac{a^2}{2}\left[\int_0^{\pi/2}(-2\cos^2\theta + 2\cos\theta)\,d\theta\right]$$

$$= \frac{a^2}{2}\left[\int_0^{\pi/2} -2\cos^2\theta d\theta + \int_0^{\pi/2} 2\cos\theta\,d\theta\right]$$



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$$= \frac{a^2}{2}\left[\left(-2\frac{\pi}{4}\right) + 2\,(\sin\theta)_0^{\pi/2}\right]$$

$$= \frac{a^2}{2}\left[-\frac{\pi}{2} + 2\left(\sin\frac{\pi}{2} - \sin 0\right)\right]$$

$$= \frac{a^2}{2}\left[-\frac{\pi}{2} + 2\right] = a^2\left(1 - \frac{\pi}{4}\right)$$ **Ans.**

Example 24. *Find by double integration, the area lying inside a cardioid r = 1 + cos θ and outside the parabola r (1 + cos θ) = 1.*

Solution. We have,

$$r = 1 + \cos\theta \quad ...(1)$$

$$r\,(1 + \cos\theta) = 1 \quad ...(2)$$

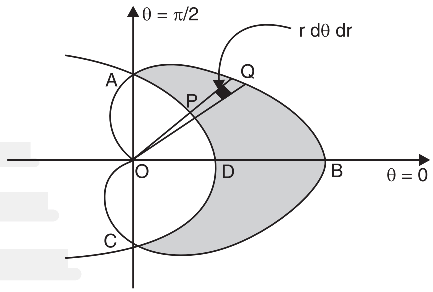
Solving (1) and (2), we get

$$(1 + \cos\theta)\,(1 + \cos\theta) = 1$$

$$(1 + \cos\theta)^2 = 1$$

$$1 + \cos\theta = 1$$

$$\cos\theta = 0 \Rightarrow \theta = \pm\frac{\pi}{2}$$



limits of r are $1 + \cos\theta$ and $\frac{1}{1+\cos\theta}$ limits of θ are $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

Required area = Area *ADCBA* (Shaded portion)

$$= \int_{-\pi/2}^{\pi/2}\int_{\frac{1}{1+\cos\theta}}^{1+\cos\theta} r\,d\theta\,dr = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\left(\frac{r^2}{2}\right)_{\frac{1}{1+\cos\theta}}^{1+\cos\theta} d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[(1+\cos\theta)^2 - \frac{1}{(1+\cos\theta)^2}\right] d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[(1+\cos^2\theta + 2\cos\theta) - \frac{1}{\left(2\cos^2\frac{\theta}{2}\right)^2}\right] d\theta$$

$$= 2\times\frac{1}{2}\int_0^{\pi/2}\left[(1+\cos^2\theta + 2\cos\theta) - \frac{1}{4}\sec^4\frac{\theta}{2}\right] d\theta$$

$$= \int_0^{\pi/2}\left[(1+\cos^2\theta + 2\cos\theta) - \frac{1}{4}\left(1+\tan^2\frac{\theta}{2}\right)\sec^2\frac{\theta}{2}\right] d\theta$$

$$= \int_0^{\pi/2}\left[\left(1 + \frac{1+\cos 2\theta}{2} + 2\cos\theta\right) - \frac{1}{4}\left(1+\tan^2\frac{\pi}{2}\right)\sec^2\frac{\pi}{2}\right] d\theta$$

$$= \int_0^{\pi/2}\left[1 + \frac{1}{2} + \frac{\cos 2\theta}{2} + 2\cos\theta - \frac{1}{4}\left(\sec^2\frac{\theta}{2} + \tan^2\frac{\theta}{2}\times\sec^2\frac{\theta}{2}\right)\right] d\theta$$

$$= \left[\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} + 2\sin\theta - \frac{1}{4}\left(2\tan\frac{\theta}{2} + \frac{2}{3}\tan^3\frac{\theta}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{\pi}{2} + \frac{\pi}{4} + 0 + 2\sin\frac{\pi}{2} - \frac{1}{2}\tan\frac{\pi}{4} - \frac{1}{6}\tan^3\frac{\pi}{4}\right] = \left[\frac{3\pi}{4} + 2 - \frac{1}{2} - \frac{1}{6}\right] = \left[\frac{3\pi}{4} + \frac{4}{3}\right]$$ **Ans.**

EXERCISE 33.6

1. Find the area of cardioid $r = a(1 + \cos\theta)$.
2. Find the area of the curve $r^2 = a^2 \cos 2\theta$.
3. Find the area enclosed by the curve $r = 2a\cos\theta$
4. Find the area enclosed by the curve $r = 3 + 2\cos\theta$.
5. Find the area enclosed by the curve

   $r^3 = a^2\cos^2\theta + b^2\sin^2\theta$.
6. Show that the area of the region included between the cardioides $r = a(1 + \cos\theta)$ and $r = a(1 - \cos\theta)$ is $\frac{a^2}{2}(3\pi - 8)$.
7. Find the area outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta)$.
8. Find the area inside the circle $r = 2a\cos\theta$ and outside the circle $r = a$.
9. Find the area inside the circle $r = 4\sin\theta$ and outside the lemniscate $r^2 = 8\cos 2\theta$.

ANSWERS

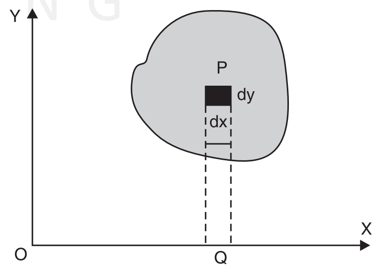
1. $\frac{3\pi a^2}{2}$	**2.** a^2	**3.** πa^2	**4.** 11π
5. $\frac{\pi}{2}(a^2 + b^2)$	**7.** $(\pi + 8)$	**8.** $2a^2\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right)$	**9.** $\left(\frac{8}{3}\pi + 4\sqrt{3} - 4\right)$

33.8 VOLUME OF SOLID BY ROTATION OF AN AREA (DOUBLE INTEGRAL)

When the area enclosed by a curve $y = f(x)$ is revolved about an axis, a solid is generated, we have to find out the volume of solid generated.

Volume of the solid generated about x-axis

$$= \int_a^b \int_{y_1(x)}^{y_2(x)} 2\pi\, PQ\, dx\, dy$$



Example 25. *Find the volume of the torus generated by revolving the circle $x^2 + y^2 = 4$ about the line $x = 3$.*

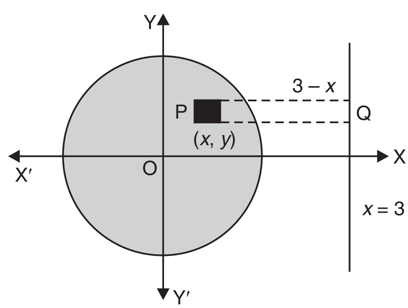
Solution. $x^2 + y^2 = 4$

$$V = \iint (2\pi\, PQ)\, dx\, dy = 2\pi \iint (3 - x)\, dx\, dy$$

$$= 2\pi \int_{-2}^{+2} dx \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} (3 - x)\, dy$$

$$= 2\pi \int_{-2}^{+2} dx \left(3y - xy\right)_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}$$

$$= 2\pi \int_{-2}^{+2} dx\, [3\sqrt{4-x^2} - x\sqrt{4-x^2} + 3\sqrt{4-x^2} - x\sqrt{4-x^2}$$



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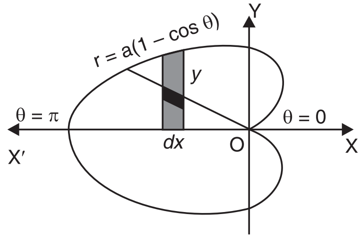
$$= 4\pi\int_{-2}^{2}[3\sqrt{4-x^2} - x\sqrt{4-x^2}]\,dx$$

$$= 4\pi\left[3\frac{x}{2}\sqrt{4-x^2} + 3\times\frac{4}{2}\sin^{-1}\frac{x}{2} + \frac{1}{3}(4-x^2)^{3/2}\right]_{-2}^{2} = 4\pi\left[6\times\frac{\pi}{2} + 6\times\frac{\pi}{2}\right] = 24\,\pi^2 \quad \textbf{Ans.}$$

Example 26. *Calculate by double integration the volume generated by the revolution of the cardioid r = a (1 – cos θ) about its axis.* (***AMIETE June 2010***)

Solution. $r = a\,(1 - \cos\theta)$



$$V = 2\pi\int\int y\,dx\,dy \Rightarrow V = 2\pi\int\int(r\,d\theta\,dr)\,y$$

$$= 2\pi\int d\theta\int r\,dr\,(r\sin\theta)$$

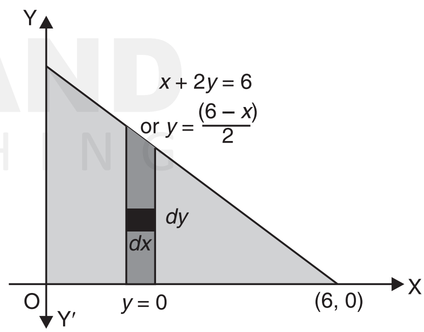
$$= 2\pi\int_0^{\pi}\sin\theta\,d\theta\int_0^{a(1-\cos\theta)} r^2\,dr$$

$$= 2\pi\int_0^{\pi}\sin\theta\,d\theta\left[\frac{r^3}{3}\right]_0^{a(1-\cos\theta)} = \frac{2\pi}{3}\int_0^{\pi}a^3\,(1-\cos\theta)^3\sin\theta\,d\theta$$

$$= \frac{2\pi a^3}{3}\left[\frac{(1-\cos\theta)^4}{4}\right]_0^{\pi} = \frac{2\pi a^3}{12}[16] = \frac{8}{3}\pi a^3 \quad \textbf{Ans.}$$

Example 27. *A pyramid is bounded by the three co-ordinate planes and the plane x + 2y + 3z = 6. Compute this volume by double integration.*

Solution. $x + 2y + 3z = 6$ ...(1)



$x = 0, y = 0, z = 0$ are co-ordinate planes.

The line of intersection of plane (1) and xy plane ($z = 0$) is

$$x + 2y = 6 \qquad ...(2)$$

The base of the pyramid may be taken to be the triangle bounded by x-axis, y-axis and the line (2).

An elementary area on the base is $dx\,dy$.

Consider the elementary rod standing on this area and having height z, where

$$3z = 6 - x - 2y \text{ or } z = \frac{6-x-2y}{3}$$

Volume of the rod = $dx\,dy$, z, Limits for z are 0 and $\frac{6-x-2y}{3}$.

Limits of y are 0 and $\frac{6-x}{2}$ and limits of x are 0 and 6.

$$\text{Required volume} = \int_0^6\int_0^{\frac{6-x}{2}} z\,dx\,dy = \int_0^6 dx\int_0^{\frac{6-x}{2}}\frac{6-x-2y}{3}\,dy$$

$$= \frac{1}{3}\int_0^6 dy\left(6y - xy - y^2\right)_0^{\frac{6-x}{2}} = \frac{1}{3}\int_0^6\left(\frac{6(6-x)}{2} - \frac{x(6-x)}{2} - \left(\frac{6-x}{2}\right)^2\right)dx$$

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$$= \frac{1}{3}\int_0^6 \left(\frac{36-6x}{2} - \frac{6x-x^2}{2} - \frac{36+x^2-12x}{4} \right) dx$$

$$= \frac{1}{12}\int_0^6 (72-12x-12x+2x^2-36-x^2+12x)\, dx$$

$$= \frac{1}{12}\int_0^6 (x^2-12x+36)\, dx = \frac{1}{12}\left[\frac{x^3}{3} - \frac{12x^2}{2} + 36x \right]_0^6 = \frac{1}{12}[72-216+216] = 6 \qquad \textbf{Ans.}$$

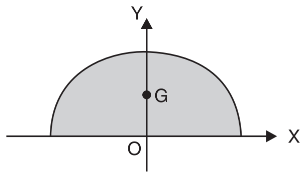
EXERCISE 33.7

1. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ by revolving area of the circle $x^2 + y^2 = a^2$.

ANSWERS

1. $\frac{4}{3}\pi a^3$

33.9 CENTRE OF GRAVITY



$$\bar{x} = \frac{\int\int \rho\, x\, dx\, dy}{\int\int \rho\, dx\, dy}, \quad \bar{y} = \frac{\int\int \rho\, y\, dx\, dy}{\int\int \rho\, dx\, dy}$$

Example 28. *Find the position of the C.G. of a semi-circular lamina of radius a if its density varies as the square of the distance from the diameter.* (*AMIETE, Dec. 2010*)

Solution. Let the bounding diameter be as the x-axis and a line perpendicular to the diameter and passing through the centre is y-axis. Equation of the circle is $x^2 + y^2 = a^2$. By symmetry $\bar{x} = 0$.

$$\bar{y} = \frac{\int\int y\, \rho\, dx\, dy}{\int\int \rho\, dx\, dy}$$

$$= \frac{\int\int (\lambda\, y^2)\, y\, dx\, dy}{\int\int (\lambda\, y^2)\, dx\, dy} = \frac{\int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} y^3\, dy}{\int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} y^2\, dy}$$

$$= \frac{\int_{-a}^{a} dx \left[\frac{y^4}{4} \right]_0^{\sqrt{a^2-x^2}}}{\int_{-a}^{a} dx \left(\frac{y^3}{3} \right)_0^{\sqrt{a^2-x^2}}} = \frac{3\int_{-a}^{a} (a^2-x^2)^2\, dx}{4\int_{-a}^{a} (a^2-x^2)^{3/2}\, dx}$$

Put $x = a \sin\theta$

$$= \frac{3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (a^2 - a^2\sin^2\theta)^2\, a\cos\theta\, d\theta}{4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (a^2 - a^2\sin^2\theta)^{3/2}\, a\cos\theta\, d\theta} = \frac{3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^5\cos^5\theta\, d\theta}{4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^4\cos^4\theta\, d\theta}$$

$$= \frac{3a}{4}\,\frac{\frac{4\times 2}{5\times 3}}{\frac{3\times 1}{4\times 2}\,\frac{\pi}{2}} = \left(\frac{3a}{4}\right)\left(\frac{8}{15}\right)\left(\frac{16}{3\pi}\right) = \frac{32a}{15\pi}$$

Hence C.G. is $\left(0, \frac{32a}{15\pi}\right)$ **Ans.**

Example 29. *Find C.G. of the area in the positive quadrant of the curve*

$$x^{2/3} + y^{2/3} = a^{2/3}.$$

Solution. For C.G. of area; $\bar{x} = \dfrac{\int\int x\, dx\, dy}{\int\int dx\, dy}$, $\bar{y} = \dfrac{\int\int y\, dx\, dy}{\int\int dx\, dy}$

$$\bar{x} = \frac{\int_0^a x\,dx \int_0^{(a^{2/3}-x^{2/3})^{3/2}} dy}{\int_0^a dx \int_0^{(a^{2/3}-x^{2/3})^{3/2}} dy} = \frac{\int_0^a x\,dx\,[y]_0^{(a^{2/3}-x^{2/3})^{3/2}}}{\int_0^a dx\,[y]_0^{(a^{2.3}-x^{2/3})^{3/2}}} \qquad [\text{Put } x = a\cos^3\theta]$$

$$= \frac{\int_0^a x\,dx\,(a^{2/3}-x^{2/3})^{3/2}}{\int_0^a dx\,(a^{2/3}-x^{2/3})^{3/2}} = \frac{\int_{\frac{\pi}{2}}^0 a\cos^3\theta\,(a^{2/3}-a^{2/3}\cos^2\theta)^{3/2}\,(-3a\cos^2\theta\sin\theta\,d\theta)}{\int_{\frac{\pi}{2}}^0 (a^{2/3}-a^{2/3}\cos^2\theta)^{3/2}\,(-3a\cos^2\theta\sin\theta\,d\theta)}$$

$$= \frac{\int_0^{\frac{\pi}{2}} 3a^3\cos^3\theta\sin^3\theta\cos^2\theta\sin\theta\,d\theta}{\int_0^{\frac{\pi}{2}} 3a^2\sin^3\theta\cos^2\theta\sin\theta\,d\theta} = \frac{a\int_0^{\frac{\pi}{2}}\sin^4\theta\cos^5\theta\,d\theta}{\int_0^{\frac{\pi}{2}}\sin^4\theta\cos^2\theta\,d\theta} = \frac{\dfrac{\left|\frac{5}{2}\right.\left|\frac{6}{2}\right.}{2\left|\frac{11}{2}\right.}\,a}{\dfrac{\left|\frac{5}{2}\right.\left|\frac{3}{2}\right.}{2\left|4\right.}}$$

$$= \frac{|3\,|4\,a}{\left|\frac{3}{2}\right.\left|\frac{11}{2}\right.} = \frac{(2)(6)\,a}{\frac{1}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\pi} = \frac{256\,a}{315\,\pi}, \quad \text{Similarly,} \quad \bar{y} = \frac{256\,a}{315\,\pi}$$

Hence, C.G. of the area is $\left(\frac{256\,a}{315\,\pi}, \frac{256\,a}{315\,\pi}\right)$.

Example 30. *Find by double integration, the centre of gravity of the area of the cardioid $r = a\,(1 + \cos\theta)$.*

Solution. Let $(\bar{x}, \bar{y})$ be the C.G. the cardioid

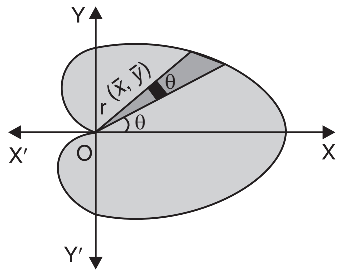
By Symmetry, $\bar{y} = 0$.

$$\bar{x} = \frac{\int\int_A x\,dx\,dy}{\int\int_A dx\,dy} = \frac{\int\int_A x\,dx\,dy}{\int\int_A dx\,dy}$$

$$= \frac{\int_{-\pi}^{\pi}\int_{0}^{a(1+\cos\theta)} (r\cos\theta)(r\,d\theta\,dr)}{\int_{-\pi}^{\pi}\int_{0}^{a(1+\cos\theta)} r\,d\theta\,dr} = \frac{\int_{-\pi}^{\pi}\cos\theta\,d\theta\int_{0}^{a(1+\cos\theta)} r^2\,dr}{\int_{-\pi}^{\pi}d\theta\int_{0}^{a(1+\cos\theta)} r\,dr}$$

$$= \frac{\int_{-\pi}^{\pi}\cos\theta\,d\theta\left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)}}{\int_{-\pi}^{\pi}d\theta\left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)}} = \frac{\int_{-\pi}^{\pi}\cos\theta\,d\theta.\frac{a^3}{3}(1+\cos\theta)^3}{\int_{-\pi}^{\pi}d\theta\,\frac{a^2}{2}(1+\cos\theta)^2}$$

$$= \frac{\frac{a^3}{3}\int_{-\pi}^{\pi}\left(2\cos^2\frac{\theta}{2}-1\right)\left(1+2\cos^2\frac{\theta}{2}-1\right)^3 d\theta}{\frac{a^2}{2}\int_{-\pi}^{\pi}\left(1+2\cos^2\frac{\theta}{2}-1\right)^2 d\theta}$$



$$= \frac{a^3}{3}\int_{-\pi}^{\pi}\left(2\cos^2\frac{\theta}{2}-1\right)\left(8\cos^6\frac{\theta}{2}\right)d\theta \div \frac{a^2}{2}\int_{-\pi}^{\pi}4\cos^4\frac{\theta}{2}\,d\theta$$

$$= \frac{8a^3}{3}\int_{-\pi}^{\pi}\left(2\cos^8\frac{\theta}{2}-\cos^6\frac{\theta}{2}\right)d\theta \div 2a^2\int_{-\pi}^{\pi}\cos^4\frac{\theta}{2}\,d\theta$$

$$= \frac{2\times 8a^3}{3}\int_{0}^{\pi}\left(2\cos^8\frac{\theta}{2}-\cos^6\frac{\theta}{2}\right)d\theta \div 4a^2\int_{0}^{\pi}\cos^4\frac{\theta}{2}\,d\theta$$

$$= \frac{16a^3}{3}\int_{0}^{\pi/2}(2\cos^8 t-\cos^6 t)(2\,dt) \div 4a^2\int_{0}^{\pi/2}\cos^4 t\,(2\,dt)$$

$$= \frac{32\,a^3}{3}\left[\frac{2\times7\times5\times3\times1}{8\times6\times4\times2}\,\frac{\pi}{2}-\frac{5\times3\times1}{6\times4\times2}\,\frac{\pi}{2}\right] \div 8a^2\left(\frac{3\times1}{4\times2}\,\frac{\pi}{2}\right)$$

$$= \frac{32a^3}{3}\left(\frac{35\pi}{128}-\frac{5\pi}{32}\right) \div 8a^2\left(\frac{3\pi}{16}\right) = \frac{32a^3}{3}\times\frac{15\pi}{128}\times\frac{16}{8a^2\times3\pi} = \frac{5a}{6}$$ **Ans.**

33.10 CENTRE OF GRAVITY OF AN ARC

Example 31. *Find the C.G. of the arc of the curve*

$x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ in the positive quadrant.

Solution. We know that, $\bar{x} = \frac{\int x\,ds}{\int ds}, \bar{y} = \frac{\int y\,ds}{\int ds}$

Now, $ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$

$$= \sqrt{\{a^2(1+\cos\theta)^2 + a^2\sin^2\theta\}}\,d\theta = a\sqrt{1+2\cos\theta+\cos^2\theta+\sin^2\theta}\,d\theta$$

$$= a\sqrt{1+2\cos\theta+1}\,d\theta = a\sqrt{2(1+\cos\theta)}\,d\theta = a\sqrt{4\cos^2\frac{\theta}{2}}\,d\theta = 2a\cos\frac{\theta}{2}\,d\theta$$

$$\bar{x} = \frac{\int x\,ds}{\int ds} = \frac{\int_0^\pi a(\theta+\sin\theta)\,2a\cos\frac{\theta}{2}\,d\theta}{\int_0^\pi 2a\cos\frac{\theta}{2}\,d\theta} = \frac{2a^2\int_0^\pi\left(2\theta\cos\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta\right)}{\left[2\sin\frac{\theta}{2}\right]_0^\pi}$$

$$= \frac{a}{2}\int_0^\pi\left[\theta\cos\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\right]d\theta = \frac{a}{2}\int_0^{\frac{\pi}{2}}(2t\cos t+2\sin t\cos^2 t)\,2\,dt$$

$$= 2a\left[t\sin t+\cos t-\frac{\cos^3 t}{3}\right]_0^{\frac{\pi}{2}} = 2a\left[\frac{\pi}{2}-1+\frac{1}{3}\right] = a\left[\pi-\frac{4}{3}\right]$$

$$\bar{y} = \frac{\int y\,ds}{\int ds} = \frac{\int_0^\pi a(1-\cos\theta)\,2a\cos\frac{\theta}{2}\,d\theta}{\int_0^\pi 2a\cos\frac{\theta}{2}\,d\theta} = \frac{a\int_0^\pi 2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta}{\int_0^\pi\cos\frac{\theta}{2}\,d\theta}$$

$$= \frac{4a\left[\sin^3\frac{\theta}{2}\right]_0^\pi}{3\left[2\sin\frac{\theta}{2}\right]_0^\pi} = \frac{4a}{3\times 2} = \frac{2a}{3}$$ Hence, C.G. of the arc is $\left[a\left(\pi-\frac{4}{3}\right),\frac{2a}{3}\right]$ **Ans.**

EXERCISE 33.8

1. Find the centre of gravity of the area bounded by the parabola $y^2 = x$ and the line $x + y = 2$.
2. Find the centroid of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$, the density at any point varying as its distance from the face $z = 0$.
3. Find the centroid of the area enclosed by the parabola $y^2 = 4\,ax$, the axis of x and latus rectum.
4. Find the centroid of the loop of curve $r^2 = a^2\cos 2\,\theta$.
5. Find the centroid of solid formed by revolving about the x-axis that part of the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which lies in the first quadrant.
6. Find the average density of the sphere of radius a whose density at a distance r from the centre of the sphere is $\rho=\rho_0\left[1+k\frac{r^3}{a^3}\right]$.
7. The density at a point on a circular lamina varies as the distance from a point O on the circumference. Show that the C.G. divides the diameter through O in the ratio 3 : 2.

ANSWERS

1. $\left(\frac{8}{5},-\frac{1}{2}\right)$ 2. $\left(\frac{1}{5},\frac{1}{5},\frac{2}{5}\right)$ 3. $\left(\frac{3a}{20},\frac{3a}{16}\right)$ 4. $\left(\frac{\pi a\sqrt{2}}{8},0\right)$
5. $\left(\frac{3a}{8},0\right)$ 6. $\rho_0\left(1+\frac{k}{2}\right)$

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33.11 TRIPLE INTEGRATION

Let a function $f(x, y, z)$ be a continuous at every point of a finite region S of three dimensional space. Consider n sub-spaces $\delta S_1, \delta S_2, \delta S_3, \ldots \delta S_n$ of the space S.

If (x_r, y_r, z_r) be a point in the rth subspace.

The limit of the sum $\sum_{r=1}^{n} f(x_r, y_r, z_r)\, \delta S_r$, as $n \to \infty$, $\delta S_r \to 0$ is known as the triple integral of $f(x, y, z)$ over the space S.

Symbolically, it is denoted by

$$\iiint_S f(x, y, z)\, dS$$

It can be calculated as $\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\; dx\, dy\, dz$. First we integrate with respect to z treating x, y as constant between the limits z_1 and z_2. The resulting expression (function of x, y) is integrated with respect to y keeping x as constant between the limits y_1 and y_2. At the end we integrate the resulting expression (function of x only) within the limits x_1 and x_2.

$$\int_{x_1 = a}^{x_2 = b} \Psi(x)\, dx \quad \boxed{\int_{y_1 = \phi_1(x)}^{y_2 = \phi_2(x)} \phi(x, y)\, dy \quad \boxed{\int_{z_1 = f_1(x, y)}^{z_2 = f_2(x, y)} f(x, y, z)\, dz}}$$

First we integrate from inner most integral w.r.t. z, then we integrate with respect to y and finally the outer most with respect to x.

But the above order of integration is immaterial provided the limits change accordingly.

Example 32. *Evaluate* $\iiint_R (x + y + z)\; dx\, dy\, dz$, *where* $R : 0 \le x \le 1,\ 1 \le y \le 2,\ 2 \le z \le 3$.

Solution. $\int_0^1 dx\int_1^2 dy\int_2^3 (x + y + z)\, dz \;=\; \int_0^1 dx\int_1^2 dy\left[\frac{(x + y + z)^2}{2}\right]_2^3$

$$= \frac{1}{2}\int_0^1 dx\int_1^2 dy\, [(x + y + 3)^2 - (x + y + 2)^2] = \frac{1}{2}\int_0^1 dx\int_1^2 (2x + 2y + 5)\,.1.dy$$

$$= \frac{1}{2}\int_0^1 dx\left[\frac{(2x + 2y + 5)^2}{4}\right]_1^2 = \frac{1}{8}\int_0^1 dx\, [(2x + 4 + 5)^2 - (2x + 2 + 5)^2]$$

$$= \frac{1}{8}\int_0^1 (4x + 16)\,.\,2\, dx = \int_0^1 (x + 4)\, dx = \left[\frac{x^2}{2} + 4x\right]_0^1 = \frac{1}{2} + 4 = \frac{9}{2}$$ **Ans.**

Example 33. *Evaluate the integral :* $\int_0^{\log 2}\int_0^{x}\int_0^{x + \log y} e^{x + y + z}\; dz\, dy\, dx.$

Solution. $\int_0^{\log 2}\int_0^{x}\int_0^{x + \log y} e^{x + y + z}\; dz\, dy\, dx.$

$$= \int_0^{\log 2} e^x dx\int_0^x e^y dy\int_0^{x + \log y} e^z dz = \int_0^{\log 2} e^x\; dx\int_0^x e^y dy (e^z)_0^{x + \log y}$$

$$= \int_0^{\log 2} e^x dx\int_0^x e^y dy\; (e^{x + \log y} - 1) = \int_0^{\log 2} e^x\; dx\int_0^x e^y dy\; (e^{\log y}\,.\, e^x - 1)$$

$$= \int_0^{\log 2} e^x dx \int_0^x e^y\,(ye^x - 1)\,dy = \int_0^{\log 2} e^x dx \left[(ye^x - 1)\,e^y - \int e^x \,.\, e^y\,dy\right]_0^x$$

$$= \int_0^{\log 2} e^x dx \left[(ye^x - 1)\,e^y - e^{x+y}\right]_0^x = \int_0^{\log 2} e^x dx\,[(xe^x - 1)\,e^x - e^{2x} + 1 + e^x]$$

$$= \int_0^{\log 2} e^x dx\,[xe^{2x} - e^x - e^{2x} + 1 + e^x] = \int_0^{\log 2} (xe^{3x} - e^{3x} + e^x)\,dx$$

$$= \left[x\frac{e^{3x}}{3} - \int 1\,.\,\frac{e^{3x}}{3}\,dx - \frac{e^{3x}}{3} + e^x\right]_0^{\log 2} = \left[\frac{x}{3}e^{3x} - \frac{e^{3x}}{9} - \frac{e^{3x}}{3} + e^x\right]_0^{\log 2}$$

$$= \frac{\log 2}{3}e^{3\log 2} - \frac{e^{3\log 2}}{9} - \frac{e^{3\log 2}}{3} + e^{\log 2} + \frac{1}{9} + \frac{1}{3} - 1$$

$$= \frac{\log 2}{3}e^{\log 2^3} - \frac{e^{\log 2^3}}{9} - \frac{e^{\log 2^3}}{3} + e^{\log 2} + \frac{1}{9} + \frac{1}{3} - 1$$

$$= \frac{8}{3}\log 2 - \frac{8}{9} - \frac{8}{3} + 2 + \frac{1}{9} + \frac{1}{3} - 1 = \frac{8}{3}\log 2 - \frac{19}{9}$$ **Ans.**

Example 34. *Evaluate* $\int_0^{\log 2} \int_0^{x} \int_0^{x+y} e^{x+y+z}\;dx\,dy\,dz.$

Solution. $I = \int_0^{\log 2} \int_0^x e^{x+y}\left[e^z\right]_0^{x+y} dx\,dy$

$$= \int_0^{\log 2}\int_0^x e^{x+y}(e^{x+y} - 1)\,dx\,dy = \int_0^{\log 2}\int_0^x \left[e^{2(x+y)} - e^{(x+y)}\right]dx\,dy$$

$$= \int_0^{\log 2}\left[e^{2x}\,.\,\frac{e^{2y}}{2} - e^x\,.\,e^y\right]_0^x dx = \int_0^{\log 2}\left(\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x\right)dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + e^x\right]_0^{\log 2} = \left[\frac{e^{4\log 2}}{8} - \frac{e^{2\log 2}}{2} - \frac{e^{2\log 2}}{4} + e^{\log 2}\right] - \left(\frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 1\right)$$

$$= \left(\frac{e^{\log 16}}{8} - \frac{e^{\log 4}}{2} - \frac{e^{\log 4}}{4} + e^{\log 2}\right) - \left(\frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 1\right)$$

$$= \left(\frac{16}{8} - \frac{4}{2} - \frac{4}{4} + 2\right) - \left(\frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 1\right) = \frac{5}{8}$$ **Ans.**

Example 35. *Evaluate* $\iiint_R (x^2 + y^2 + z^2)\,dx\,dy\,dz$ *where R denotes the region bounded by* $x = 0,\ y = 0,\ z = 0$ *and* $x + y + z = a,\ (a > 0)$

Solution. $\iiint_R (x^2 + y^2 + z^2)\,dx\,dy\,dz$

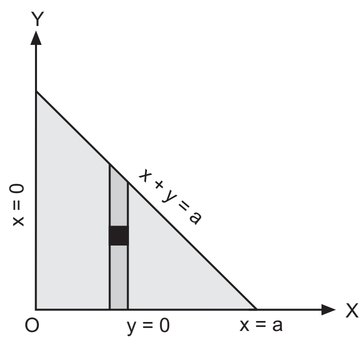
$x + y + z = a$ or $z = a - x - y$

Upper limit of $z = a - x - y$

On x-y plane, $x + y + z = a$ becomes $x + y = a$ as shown in the figure.

Upper limit of $y = a - x$

Upper limit of $x = a$



$$= \int_{x=0}^{a} dx \int_{y=0}^{a-x} dy \int_{z=0}^{a-x-y} (x^2 + y^2 + z^2)\, dz$$

$$= \int_0^a dx \int_0^{a-x} dy \left(x^2 z + y^2 z + \frac{z^3}{3} \right)_0^{a-x-y}$$

$$= \int_0^a dx \int_0^{a-x} dy \left[x^2(a-x-y) + y^2(a-x-y) + \frac{(a-x-y)^3}{3} \right]$$

$$= \int_0^a dx \int_0^{a-x} \left[x^2(a-x) - x^2 y + (a-x)\, y^2 - y^3 + \frac{(a-x-y)^3}{3} \right] dy$$

$$= \int_0^a dx \left[x^2(a-x)\, y - \frac{x^2 y^2}{2} + (a-x)\frac{y^3}{3} - \frac{y^4}{4} - \frac{(a-x-y)^4}{12} \right]_0^{a-x}$$

$$= \int_0^a dx \left[x^2(a-x)^2 - \frac{x^2}{2}(a-x)^2 + (a-x)\frac{(a-x)^3}{3} - \frac{(a-x)^4}{4} + \frac{(a-x)^4}{12} \right]$$

$$= \int_0^a \left[\frac{x^2}{2}(a-x)^2 + \frac{(a-x)^4}{6} \right] dx = \int_0^a \left[\frac{1}{2}(a^2 x^2 - 2a x^3 + x^4) + \frac{(a-x)^4}{6} \right] dx$$

$$= \left[\frac{1}{2} a^2 \frac{x^3}{3} - \frac{a x^4}{4} + \frac{x^5}{10} - \frac{(a-x)^5}{30} \right]_0^a = \frac{a^5}{6} - \frac{a^5}{4} + \frac{a^5}{10} + \frac{a^5}{30} = \frac{a^5}{20}$$ **Ans.**

Example 36. *Compute* $\iiint \frac{dx\, dy\, dz}{(x+y+z+1)^3}$ *if the region of integration is bounded by the coordinate planes and the plane* $x + y + z = 1$. (*M.U., II Semester 2007, 2006*)

Solution. Let the given region be R, then R is expressed as (*A.M.I.E.T.E., June 2017*)

$$0 \le z \le 1 - x - y, \quad 0 \le y \le 1 - x, \quad 0 \le x \le 1.$$

$$\iiint_R \frac{dx\, dy\, dz}{(x+y+z+1)^3} = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \frac{dz}{(x+y+z+1)^3}$$

$$= \int_0^1 dx \int_0^{1-x} dy \left[\frac{1}{-2(x+y+z+1)^2} \right]_0^{1-x-y}$$

$$= -\frac{1}{2} \int_0^1 dx \int_0^{1-x} dy \left[\frac{1}{(x+y+1-x-y+1)^2} - \frac{1}{(x+y+1)^2} \right]$$

$$= -\frac{1}{2} \int_0^1 dx \int_0^{1-x} \left[\frac{1}{4} - \frac{1}{(x+y+1)^2} \right] dy = -\frac{1}{2} \int_0^1 dx \left[\frac{y}{4} + \frac{1}{x+y+1} \right]_0^{1-x}$$

$$= -\frac{1}{2} \int_0^1 dx \left[\frac{1-x}{4} + \frac{1}{x+1+1-x} - \frac{1}{x+1} \right] = -\frac{1}{2} \int_0^1 \left[\frac{1-x}{4} + \frac{1}{2} - \frac{1}{x+1} \right] dx$$

$$= -\frac{1}{2} \left[-\frac{(1-x)^2}{8} + \frac{x}{2} - \log(x+1) \right]_0^1 = -\frac{1}{2} \left[\frac{1}{2} - \log 2 + \frac{1}{8} \right] = -\frac{1}{2} \left[\frac{5}{8} - \log 2 \right]$$

$$= \frac{1}{2} \log 2 - \frac{5}{16}$$ **Ans.**

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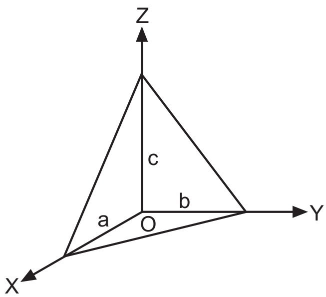
Example 37. *Evaluate* $\iiint x^2 yz \; dx\,dy\,dz$ *throughout the volume bounded by the planes* $x = 0,$

$$y = 0,\ z = 0,\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1.$$

Solution. Here, we have

$$I = \iiint x^2 yz \; dx\,dy\,dz \qquad ...(1)$$



Putting $x = au, \quad y = bv, \quad z = cw$

$dx = a\,du, \quad dy = b\,dv, \quad dz = c\,dw$ in (1), we get

$$I = \iiint a^2bc\,u^2\,v\,w\,a\,bc\,du\,dv\,dw$$

$$u + v + w = 1$$

Limits are for $u = 0, 1$ for $v = 0, 1 - u$ and for $w = 0, 1 - u - v$

$$I = \int_{u=o}^{1} \int_{v=0}^{1-u} \int_{w=0}^{1-u-v} a^3b^2c^2\,u^2vw\,du\,dv\,dw$$

$$= \int_0^1 \int_0^{1-u} a^3b^2c^2\,u^2v\left[\frac{w^2}{2}\right]_0^{1-u-v} du\,dv = \frac{a^3b^2c^2}{2}\int_0^1 \int_0^{1-u} u^2v(1-u-v)^2\,du\,dv$$

$$= \frac{a^3b^2c^2}{2}\int_0^1 \int_0^{1-u} u^2v\left[(1-u)^2 - 2(1-u)v + v^2\right] du\,dv$$

$$= \frac{a^3b^2c^2}{2}\int_0^1 \int_0^{1-u} u^2\left[(1-u)^2 v - 2\,(1-u)\,v^2 + v^3\right]\,du\,dv$$

$$= \frac{a^3b^2c^2}{2}\int_0^1 u^2\left[(1-u)^2\frac{v^2}{2} - 2(1-u)\frac{v^3}{3} + \frac{v^4}{4}\right]_0^{1-u} du$$

$$= \frac{a^3b^2c^2}{2}\int_0^1 u^2\left[\frac{(1-u)^4}{2} - \frac{2(1-u)^4}{3} + \frac{(1-u)^4}{4}\right] du = \frac{a^3b^2c^2}{2}\int_0^1 \frac{u^2(1-u)^4}{12}\,du$$

$$= \frac{a^3b^2c^2}{24}\int_0^1 u^{3-1}\,(1-u)^{5-1}\,du = \frac{a^3b^2c^2}{24}\,\beta(3,5) = \frac{a^3b^2c^2}{24}\cdot\frac{\lceil 3\,\lceil 5}{\lceil 8}$$

$$= \frac{a^3b^2c^2}{24}\cdot\left(\frac{2!\,4!}{7!}\right) = \frac{a^3\,b^2\,c^2}{2520}.$$ **Ans.**

33.12 INTEGRATION BY CHANGE OF CARTESIAN COORDINATES INTO SPHERICAL COORDINATES

Sometime it becomes easy to integrate by changing the cartesian coordinates into spherical coordinates.

The relations between the cartesian and spherical polar co-ordinates of a point are given by the relations

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$
$$dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi$$
$$= r^2\sin\theta\,dr\,d\theta\,d\phi$$

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Note. 1. Spherical coordinates are very useful if the expression $x^2 + y^2 + z^2$ is involved in the problem.

2. In a sphere $x^2 + y^2 + z^2 = a^2$ the limits of r are 0 and a and limits of θ are 0, π and that of ϕ are 0 and 2π.

Example 38. *Evaluate the integral* $\iiint (x^2 + y^2 + z^2)\ dx\,dy\,dz$ *taken over the volume enclosed by the sphere* $x^2 + y^2 + z^2 = 1$.

Solution. Let us convert the given integral into spherical polar co-ordinates. By putting

$$x = r\sin\theta\cos\phi\ ;\quad y = r\sin\theta\sin\phi\ ;\quad z = r\cos\theta$$

$$\iiint (x^2 + y^2 + z^2)\ dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^1 r^2(r^2\sin\theta\,dr\,d\theta\,d\phi)$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\,d\theta \int_0^1 r^4 dr = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\,d\theta \left(\frac{r^5}{5}\right)_0^1 = \frac{1}{5}\int_0^{2\pi} d\phi\,[-\cos\theta]_0^{\pi} = \frac{2}{5}\int_0^{2\pi} d\phi$$

$$= \frac{2}{5}(\phi)_0^{2\pi} = \frac{4\pi}{5}$$ **Ans.**

Example 39. *Evaluate* $\iiint (x^2 + y^2 + z^2)\ dx\,dy\,dz$ *over the first octant of the sphere* $x^2 + y^2 + z^2 = a^2$. (*M.U. II Semester 2007*)

Solution. Here, we have

$$I = \iiint (x^2 + y^2 + z^2)\ dx\,dy\,dz \qquad ...(1)$$

Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$ in (1), we get

Limits of r are 0, a for θ are 0, $\frac{\pi}{2}$ for ϕ are 0, $\frac{\pi}{2}$.

$$I = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^a r^2 . r^2\sin\theta\,dr\,d\theta\,d\phi = \int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} \sin\theta\,d\theta \int_0^a r^4\,dr$$

$$\left(\begin{array}{c} x^2 + y^2 + z^2 = r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta \\ = r^2\sin^2\theta + r^2\cos^2\theta = r^2 \end{array}\right)$$

$$= [\phi]_0^{\pi/2}\,[-\cos\theta]_0^{\pi/2}\left[\frac{r^5}{5}\right]_0^a = \frac{\pi}{2}.(1).\frac{a^5}{5} = \pi.\frac{a^5}{10}.$$ **Ans.**

Example 40. *Evaluate* $\iiint \dfrac{dx\,dy\,dz}{x^2 + y^2 + z^2}$ *throughout the volume of the sphere* $x^2 + y^2 + z^2 = a^2$.

Solution. Here, we have

$$I = \iiint \frac{dx\,dy\,dz}{x^2 + y^2 + z^2} \qquad ...(1)$$

Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$ in (1), we get

The limits of r are 0 and a, for θ are 0 and $\frac{\pi}{2}$ for ϕ are 0 and $\frac{\pi}{2}$ in first octant.

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$$I = 8\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a \frac{r^2 \sin\theta \, dr \, d\theta \, d\phi}{r^2} \qquad \text{[Sphere } x^2 + y^2 + z^2 \text{ lies in 8 quadrants]}$$

$$I = 8\int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta \int_0^a dr = 8[\phi]_0^{\pi/2} [-\cos\theta]_0^{\pi/2} [r]_0^a = 8\left(\frac{\pi}{2} - 0\right)(0+1)(a+0)$$

$$= 8\frac{\pi}{2}.1.a = 4\pi a$$

Ans.

EXERCISE 33.9

Evaluate the following:

1. $\int_{-1}^{1} \int_{-2}^{2} \int_{-3}^{3} dx \, dy \, dz$ (*M.U., II Semester 2002*)

2. $\int_0^4 \int_0^x \int_0^{x+y} z \, dz \, dy \, dx$ (*R.G.P.V. Bhopal I Sem. 2003*)

3. $\int_1^2 \int_0^1 \int_{-1}^1 (x^2 + y^2 + z^2) \, dx \, dy \, dz$

4. $\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2) \, dz \, dy \, dx$ (*AMIETE, June 2006*)

5. $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x - y + z) \, dx \, dy \, dz$ (*AMIETE, Summer 2004*)

6. $\iiint_R (x - y - z) \, dx \, dy \, dz$, where $R : 1 \le x \le 2;\ 2 \le y \le 3;\ 1 \le z \le 3$

7. $\int_0^2 \int_1^3 \int_1^2 xy^2 z \, dx \, dy \, dz$ (*AMIETE, Dec. 2007*)

8. $\int_0^1 dx \int_0^2 dy \int_1^2 x^2 \, yz \, dz$

9. $\iiint x^2 yz \, dx \, dy \, dz$ throughout the volume bounded by $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$. (*M.U. II Semester, 2003*)

10. $\int_0^1 \int_0^{1-x} \int_0^{1-x^2-y^2} dz \, dy \, dx$

11. $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz \, dx \, dy$

12. $\iiint_T y \, dx \, dy \, dz$, where T is the region bounded by the surfaces $x = y^2$, $x = y + 2$, $4z = x^2 + y^2$ and $z = y + 3$. (*AMIETE Dec. 2008*)

13. $\int_0^2 \int_0^x \int_0^{2x+2y} e^{x+y+z} \, dz \, dy \, dx$ (M.U. II Sem., 2003)

14. $\iiint (x + y + z) \, dx \, dy \, dz$ over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

15. $\int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2\, dx\, dy\, dz$

16. $\int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx$

17. $\int_{-1}^{1} \int_0^{z} \int_{x-z}^{x+z} (x+y+z)\, dz\, dx\, dy$ (*M.U. II Semester, 2000, 02*)

18. $\int_0^2 \int_0^y \int_{x-y}^{x+y} (x+y+z)\, dx\, dy\, dz$ (*M.U. II Semester 2004*)

19. $\iiint \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\, dx\, dy\, dz$ throughout the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

20. $\iiint \sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}\, dx\, dy\, dz$ over the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

21. $\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$ throughout the volume of the tetrahedron $x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y+z \leq 1$.

22. $\iiint \frac{dx\, dy\, dz}{\sqrt{1-x^2-y^2-z^2}}$ taken throughout the volume of the sphere $x^2+y^2+z^2=1$, lying in the first octant.

23. $\int_0^{\pi} 2d\theta \int_0^{a(1+\cos\theta)} r\, dr \int_0^h \left[1-\frac{r}{a(1+\cos\theta)}\right] dz$

24. $\int_0^{\pi/2} \int_0^{a\sin\theta} \int_0^{(a^2-r^2)/a} r\, d\theta\, dr\, dz$

25. $\iiint z^2\, dx\, dy\, dz$ over the volume common to the sphere $x^2+y^2+z^2=a^2$ and the cylinder $x^2+y^2+z^2=ax$.

26. $\iiint_V \frac{dx\, dy\, dz}{(1+x^2+y^2+z^2)^2}$ where V is the volume in the first octant

27. $\iiint \frac{dx\, dy\, dz}{(x^2+y^2+z^2)^{3/2}}$ over the volume bounded by the spheres $x^2+y^2+z^2=16$ and $x^2+y^2+z^2=25$. (*M.U. II Semester, 2001, 03*)$\pi \log (5/4)$

28. $\iiint_T z^2\, dx\, dy\, dz$ over the volume bounded by the cylinder $x^2+y^2=a^2$ and the paraboloid $x^2+y^2=z$ and the plane $z=0$.

ANSWERS

1. 48	**2.** 70	**3.** 6	**4.** 1
5. 0	**6.** 2	**7.** 26	**8.** 1
9. $\frac{1}{2520}$	**10.** $\frac{1}{3}$	**11.** $\frac{1}{2}(e^2-8e+13)$	
13. $\frac{1}{3}\left[\frac{e^{12}}{6}-\frac{e^6}{3}-\frac{1}{6}+\frac{1}{3}\right]-\frac{1}{2}[e^4-1]+[e^2-1]$		**14.**	$\frac{1}{8}$

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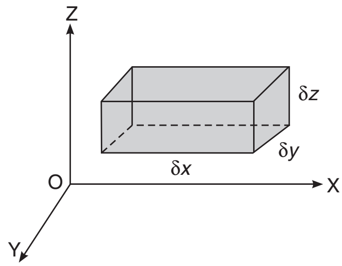
15. $\frac{a^5}{60}$	**16.** $8\sqrt{2}\pi$	**17.** 0	**18.** 16
19. $\frac{\pi^2}{4}abc$	**20.** $\frac{4\pi}{3}abc$	**21.** $\frac{1}{(l+m+n)}\cdot\frac{\lceil l \lceil m \lceil n}{\lceil l+m+n}$	**22.** $\frac{\pi^2}{8}$
23. $\frac{\pi a^2}{2}h$	**24.** $\frac{5a^3}{64}$	**25.** $\frac{2a^5\pi}{15}$	**26.** $\frac{\pi^2}{8}$
27. $4\pi \log (5/4)$	**28.** $\frac{\pi a^8}{12}$		

33.13 VOLUME = $\iiint dx\, dy\, dz$.



The elementary volume δv is $\delta x \,.\, \delta y \,.\, \delta z$ and therefore the volume of the whole solid is obtained by evaluating the triple integral.

$$\delta V = \delta x\, \delta y\, \delta z$$

$$\boxed{V = \iiint dx\, dy\, dz.}$$

Note : (*i*) Mass = volume × density = $\iiint \rho\, dx\, dy\, dz$

if ρ is the density.

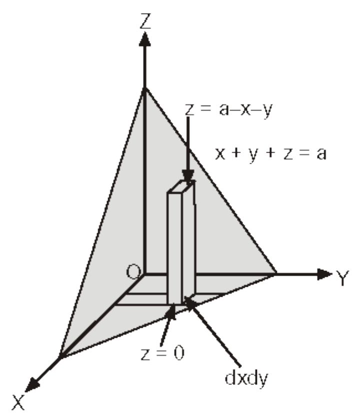
(*ii*) In cylindrical co-ordinates, we have $\boxed{V = \iiint_V r\, dr\, d\varphi\, dz}$

(*iii*) In spherical polar co-ordinates, we have $\boxed{V = \iiint_V r^2 \sin\theta dr\, d\theta\, d\varphi}$

Example 41. *Find the volume of the tetrahedron bounded by the planes x = 0, y = 0, z = 0 and x + y + z = a.* **(*A.M.I.E.T.E. June 2016, M.U. II Semester 2005*)**

Solution. Here, we have a solid which is bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = a$ planes.

The limits of z are 0 and $a - x - y$, the limits of y are 0 and $a - x$, the limits of x are 0 and a.



$$V = \int_{x=0}^{a} \int_{y=0}^{a-x} \int_{z=0}^{a-x-y} dx\, dy\, dz$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} [z]_0^{a-x-y}\, dx\, dy$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} (a-x-y)\, dy\, dx$$

$$= \int_{x=0}^{a} \left[ay - xy - \frac{y^2}{2}\right]_0^{a-x} dx$$

$$= \int_0^a \left[a(a-x) - x(a-x) - \frac{(a-x)^2}{2}\right] dx$$

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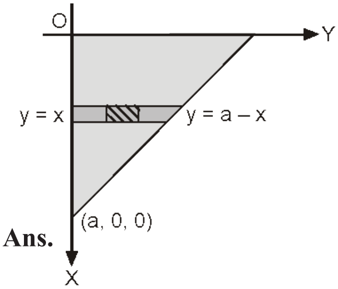
$$= \int_0^a \left[a^2 - ax - ax + x^2 - \frac{a^2}{2} + ax - \frac{x^2}{2}\right] dx$$

$$= \int_0^a \left(\frac{a^2}{2} - ax + \frac{x^2}{2}\right) dx$$

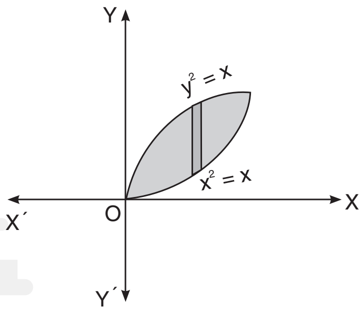
$$= \left[\frac{a^2}{2} . x - \frac{ax^2}{2} + \frac{x^3}{6}\right]_0^a = a^3\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) = \frac{a^3}{6}. \quad \textbf{Ans.}$$



Example 42. *Find the volume of the cylindrical column standing on the area common to the parabolas $y^2 = x$, $x^2 = y$ and cut off by the surface $z = 12 + y - x^2$.*

(***U.P. II Sem. Summer 2001***)

Solution. We have,



$$y^2 = x$$
$$x^2 = y$$
$$z = 12 + y - x^2$$

$$V = \int_0^1 dx \int_{x^2}^{\sqrt{x}} dy \int_0^{12+y-x^2} dz$$

$$= \int_0^1 dx \int_{x^2}^{\sqrt{x}} (12 + y - x^2)\, dy$$

$$= \int_0^1 dx \left(12y + \frac{y^2}{2} - x^2 y\right)_{x^2}^{\sqrt{x}} = \int_0^1 \left(12\sqrt{x} + \frac{x}{2} - x^{5/2} - 12x^2 - \frac{x^4}{2} + x^4\right) dx$$

$$= \left[\frac{2}{3} \times 12x^{3/2} + \frac{x^2}{4} - \frac{2}{7}x^{7/2} - 4x^3 - \frac{x^5}{10} + \frac{x^5}{5}\right]_0^1$$

$$= 8 + \frac{1}{4} - \frac{2}{7} - 4 - \frac{1}{10} + \frac{1}{5} = 4 + \frac{1}{4} - \frac{2}{7} - \frac{1}{10} + \frac{1}{5} = \frac{560 + 35 - 40 - 14 + 28}{140} = \frac{569}{140} \quad \textbf{Ans.}$$

Example 43. *A triangular prism is formed by planes whose equations are $ay = bx$, $y = 0$ and $x = a$. Find the volume of the prism between the planes $z = 0$ and surface $z = c + xy$.*

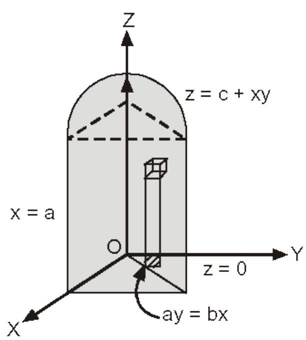
(*M.U. II Semester 2000; U.P., Ist Semester, 2009 (C.O) 2003*)

Solution. Required volume $= \int_0^a \int_0^{\frac{bx}{a}} \int_0^{c+xy} dz\, dy\, dx$

$$= \int_0^a \int_0^{\frac{bx}{a}} (c + xy)\, dy\, dx$$

$$= \int_0^a \left(cy + \frac{xy^2}{2}\right)_0^{\frac{bx}{a}} dx$$

$$= \int_0^a \left(\frac{cbx}{a} + \frac{b^2}{2a^2} x^3\right) dx$$



$$= \frac{bc}{a}\left(\frac{x^2}{2}\right)_0^a + \frac{b^2}{2a^2}\left(\frac{x^4}{4}\right)_0^a$$

$$= \frac{abc}{2} + \frac{b^2a^2}{8} = \frac{ab}{8}(4c + ab) \qquad \textbf{Ans.}$$

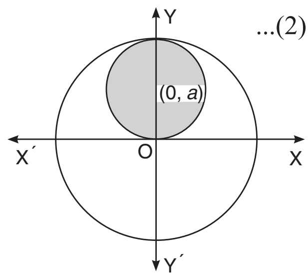
33.14 VOLUME OF SOLID BOUNDED BY SPHERE OR BY CYLINDER

We use spherical coordinates (r, θ, ϕ) and the cylindrical coordinates are (ρ, ϕ, z) and the relations are $x = \rho \cos \phi$, $y = \rho \sin \phi$.

Example 44. *Find the volume of a solid bounded by the spherical surface $x^2 + y^2 + z^2 = 4a^2$ and the cylinder $x^2 + y^2 - 2ay = 0$.*

Solution. $x^2 + y^2 + z^2 = 4a^2$ ...(1)

$x^2 + y^2 - 2ay = 0$ ...(2)



Considering the section in the positive quadrant of the *xy*-plane and taking *z* to be positive (that is volume above the *xy*-plane) and changing to polar co-ordinates, (1) becomes

$$r^2 + z^2 = 4a^2 \quad \Rightarrow \quad z^2 = 4a^2 - r^2$$

$$\therefore \qquad z = \sqrt{4a^2 - r^2}$$

(2) becomes $r^2 - 2ar \sin \theta = 0 \Rightarrow r = 2a \sin \theta$

$$\text{Volume} = \iiint dx\, dy\, dz$$

$$= 4\int_0^{\pi/2} d\theta \int_0^{2a\sin\theta} r\, dr \int_0^{\sqrt{4a^2 - r^2}} dz \qquad \text{(Cylindrical coordinates)}$$

$$= 4\int_0^{\pi/2} d\theta \int_0^{2a\sin\theta} r\, dr [z]_0^{\sqrt{4a^2 - r^2}} = 4\int_0^{\pi/2} d\theta \int_0^{2a\sin\theta} r\, dr \,.\sqrt{4a^2 - r^2}$$

$$= 4\int_0^{\pi/2} d\theta \left[-\frac{1}{3}(4a^2 - r^2)^{3/2}\right]_0^{2a\sin\theta} = \frac{4}{3}\int_0^{\pi/2} \left[-(4a^2 - 4a^2\sin^2\theta)^{3/2} + 8a^3\right] d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2} (-8a^3\cos^3\theta + 8a^3)\, d\theta = \frac{8 \times 4a^3}{3}\int_0^{\pi/2} (1 - \cos^3\theta)\, d\theta$$

$$= \frac{32a^3}{3}\int_0^{\pi/2} \left(1 - \frac{1}{4}\cos 3\theta - \frac{3}{4}\cos\theta\right) d\theta$$

$$= \frac{32a^3}{3}\left[\theta - \frac{1}{12}\sin 3\theta - \frac{3}{4}\sin\theta\right]_0^{\pi/2} = \frac{32a^3}{3}\left(\frac{\pi}{2} + \frac{1}{12} - \frac{3}{4}\right) = \frac{32a^3}{3}\left[\frac{\pi}{2} - \frac{2}{3}\right] \qquad \textbf{Ans.}$$

Example 45. *Find the volume enclosed by the solid*

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1$$

Solution. The equation of the solid is

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1$$

Putting $\quad \left(\frac{x}{a}\right)^{1/3} = u \quad \Rightarrow \quad x = a\,u^3 \quad \Rightarrow \quad d\,x = 3\,au^2\,du$

$$\left(\frac{y}{b}\right)^{1/3} = v \quad \Rightarrow \quad y = b\,v^3 \quad \Rightarrow \quad d\,y = 3\,bv^2\,dv$$

$$\left(\frac{z}{c}\right)^{1/3} = w \quad \Rightarrow \quad z = c\,w^3 \quad \Rightarrow \quad d\,z = 3\,c\,w^2\,dw$$

The equation of the solid becomes

$$u^2 + v^2 + w^2 = 1 \qquad ...(1)$$

$$V = \iiint dx\,dy\,dz \qquad ...(2)$$

On putting the values of *dx, dy* and *dz* in (2), we get

$$V = \iiint 27abc\,u^2v^2w^2du\,dv\,dw \qquad ...(3)$$

(1) represents a sphere.

Let us use spherical coordinates.

$$u = r\sin\theta\cos\phi, \qquad v = r\sin\theta\sin\phi,$$
$$w = r\cos\theta, \qquad du\,dv\,dw = r^2\sin\theta\,dr\,d\theta\,d\phi$$

On substituting spherical coordinates in (3), we have

$$V = 27abc\,.\,8\int_{r=0}^{1}\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} r^2\sin^2\theta\cos^2\phi\,.\,r^2\sin^2\theta\sin^2\phi\,.\,r^2\cos^2\theta\,.\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 216\,abc\int_{r=0}^{1} r^8\,dr\int_{\phi=0}^{\pi/2}\sin^2\phi\cos^2\phi\,d\phi\int_{\theta=0}^{\pi/2}\sin^5\theta\cos^2\theta\,d\theta$$

$$= 216\,abc\left[\frac{r^9}{9}\right]_0^1 . \left(\frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{2\,\Gamma 3}\right)\left(\frac{\Gamma 3\,\Gamma\frac{3}{2}}{2\,\Gamma\frac{9}{2}}\right) = 24\,abc\,.\,\frac{1}{2}\,.\,\frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{\Gamma 3}\,.\,\frac{1}{2}\,.\,\frac{\Gamma 3\,\Gamma\frac{3}{2}}{\Gamma\frac{9}{2}}$$

$$= 6\,abc\,.\,\frac{\left[\left(\frac{1}{2}\right)\Gamma\frac{1}{2}\right]^2}{2\,!}\,.\,\frac{2!\,\Gamma\frac{3}{2}}{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\frac{3}{2}\,\Gamma\frac{3}{2}} = 6\,abc\,.\,\frac{1}{4}\,.\,\pi\,\frac{1}{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)} = \frac{4}{35}\,abc\,\pi \qquad \textbf{Ans.}$$

Example 46. *Find the volume bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the cone $x^2 + y^2 = z^2$.*

Solution. The equation of the sphere is $x^2 + y^2 + z^2 = a^2$ ...(1)

and that of the cone is $\quad x^2 + y^2 = z^2$ ...(2)

In polar coordinates $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

The equation (1) in polar co-ordinates is

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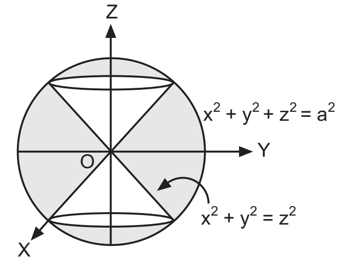

$(r \sin\theta \cos\phi)^2 + (r \sin\theta \sin\phi)^2 + (r\cos\theta)^2 = a^2$

$\Rightarrow\; r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi + r^2\cos^2\theta = a^2$

$\Rightarrow\; r^2 \sin^2\theta (\cos^2\phi + \sin^2\phi) + r^2\cos^2\theta = a^2$

$\Rightarrow\; r^2 \sin^2\theta + r^2\cos^2\theta = a^2$

$\Rightarrow\; r^2 (\sin^2\theta + \cos^2\theta) = a^2$

$\Rightarrow\; r^2 = a^2 \;\Rightarrow\; r = a$

The equation (2) in polar co-ordinates is

$(r \sin\theta \cos\phi)^2 + (r \sin\theta \sin\phi)^2 = (r\cos\theta)^2$

$\Rightarrow\; r^2 \sin^2\theta (\cos^2\phi + \sin^2\phi) = r^2\cos^2\theta \;\Rightarrow$
$r^2\sin^2\theta = r^2\cos^2\theta$

$\Rightarrow\; \tan^2\theta = 1 \;\Rightarrow\; \tan\theta = 1 \Rightarrow \quad \theta = \pm\frac{\pi}{4}$

Thus equations (1) and (2) in polar coordinates are respectively,

$$r = a \qquad \text{and} \qquad \theta = \pm\frac{\pi}{4}$$

The volume in the first octant is one fourth only.

Limits in the first octant : r varies 0 to a, θ from 0 to $\frac{\pi}{4}$ and ϕ from 0 to $\frac{\pi}{2}$.

The required volume lies between $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 = z^2$.

$$V = 4\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^a r^2 \sin\theta\, dr\, d\theta\, d\phi = 4\int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{4}} \sin\theta\, d\theta \left[\frac{r^3}{3}\right]_0^a$$

$$= 4\int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{4}} \sin\theta\, d\theta \,.\, \frac{a^3}{3} = \frac{4a^3}{3}\int_0^{\frac{\pi}{2}} d\phi \left[-\cos\theta\right]_0^{\frac{\pi}{4}} = \frac{4a^3}{3}(\phi)_0^{\frac{\pi}{2}}\left[-\frac{1}{\sqrt{2}} + 1\right]$$

$$= \frac{2}{3}\pi a^3 \left(1 - \frac{1}{\sqrt{2}}\right)$$ **Ans.**

33.15 VOLUME OF SOLID BOUNDED BY CYLINDER OR CONE

We use cylindrical coordinates (r, θ, z).

Example 47. *Find the volume of the solid bounded by the parabolic $y^2 + z^2 = 4x$ and the plane $x = 5$.*

Solution. $y^2 + z^2 = 4x$, $x = 5$

$$V = \int_0^5 dx \int_{-2\sqrt{x}}^{2\sqrt{x}} dy \int_{-\sqrt{4x-y^2}}^{\sqrt{4x-y^2}} dz = 4\int_0^5 dx \int_0^{2\sqrt{x}} dy \int_0^{\sqrt{4x-y^2}} dz$$

$$= 4\int_0^5 dx \int_0^{2\sqrt{x}} dy\, [z]_0^{\sqrt{4x-y^2}} = 4\int_0^5 dx \int_0^{2\sqrt{x}} dy\, \sqrt{4x-y^2}$$

$$= 4\int_0^5 dx \left[\frac{y}{2}\sqrt{4x-y^2} + \frac{4x}{2}\sin^{-1}\frac{y}{2\sqrt{x}}\right]_0^{2\sqrt{x}}$$

$$= 4\int_0^5 \left[0 + 2x\left(\frac{\pi}{2}\right)\right] dx = 4\pi\int_0^5 x\, dx = 4\pi\left[\frac{x^2}{2}\right]_0^5 = 50\pi$$ **Ans.**

Example 48. *Calculate the volume of the solid bounded by the following surfaces:*

$$z = 0, \quad x^2 + y^2 = 1, \quad x + y + z = 3$$

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Solution. $x^2 + y^2 = 1$ ...(1)

$x + y + z = 3$ ...(2)

$z = 0$ ...(3)

Required Volume $= \iiint dx\, dy\, dz = \iint dx\, dy \left[z\right]_0^{3-x-y} = \iint (3 - x - y)\, dx\, dy$

On putting $x = r\cos\theta$, $y = r\sin\theta$, $dx\, dy = r\, d\theta\, dr$, we get

$$= \iint (3 - r\cos\theta - r\sin\theta)\, r\, d\theta\, dr = \int_0^{2\pi} d\theta \int_0^1 (3r - r^2\cos\theta - r^2\sin\theta)\, dr$$

$$= \int_0^{2\pi} d\theta \left(\frac{3r^2}{2} - \frac{r^3}{3}\cos\theta - \frac{r^3}{3}\sin\theta\right)_0^1 = \int_0^{2\pi} \left(\frac{3}{2} - \frac{1}{3}\cos\theta - \frac{1}{3}\sin\theta\right) d\theta$$

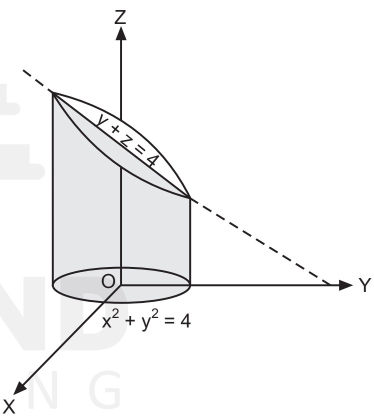
$$= \left[\frac{3}{2}\theta - \frac{1}{3}\sin\theta + \frac{1}{3}\cos\theta\right]_0^{2\pi} = 3\pi - \frac{1}{3}\sin 2\pi + \frac{1}{3}\cos 2\pi - \frac{1}{3} = 3\pi$$ **Ans.**

Example 49. *Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.* (***AMIETE Dec 2015***)

Solution. $x^2 + y^2 = 4 \Rightarrow y = \pm\sqrt{4 - x^2}$

$y + z = 4 \Rightarrow z = 4 - y$ and $z = 0$

x varies from -2 to $+2$.



$$V = \iiint dx\, dy\, dz = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy \int_0^{4-y} dz$$

$$= \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy \left[z\right]_0^{4-y}$$

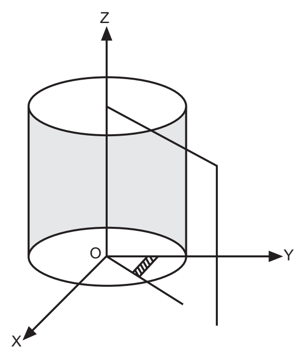
$$= \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\, (4 - y)$$

$$= \int_{-2}^{2} dx \left[4y - \frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}$$

$$= \int_{-2}^{2} dx \left[4\sqrt{4 - x^2} - \frac{1}{2}(4 - x^2) + 4\sqrt{4 - x^2} + \frac{1}{2}(4 - x^2)\right]$$

$$= 8\int_{-2}^{2} \sqrt{4 - x^2}\, dx = 8\left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{-2}^{2} = 16\pi$$ **Ans.**

Example 50. *Find the volume in the first octant bounded by the cylinder $x^2 + y^2 = 2$ and the planes $z = x + y$, $y = x$, $z = 0$ and $x = 0$.* (*M.U. II Semester 2005*)



Solution. Here, we have the solid bounded by

$x^2 + y^2 = 2$ (cylinder)

(or $r^2 = 2$)

$z = x + y \Rightarrow z = r(\cos\theta + \sin\theta)$ (plane)

$y = x \Rightarrow r\sin\theta = r\cos\theta$ (plane)

$\Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

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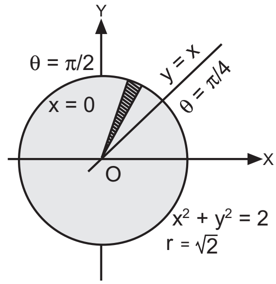
$x = 0 \Rightarrow r\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$



z varies from 0 to $r(\cos\theta + \sin\theta)$

r varies from 0 to $\sqrt{2}$

θ varies from $\frac{\pi}{4}$ to $\frac{\pi}{2}$

$$\therefore \quad V = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{\sqrt{2}}\int_{z=0}^{r(\cos\theta+\sin\theta)} dz\,dr\,d\theta$$

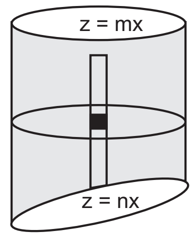
$$= \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{\sqrt{2}} r\left[z\right]_0^{r(\cos\theta+\sin\theta)} dr\,d\theta = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{\sqrt{2}} r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$

$$= \int_{\theta=\pi/4}^{\pi/2}(\cos\theta+\sin\theta)\left[\frac{r^3}{3}\right]_0^{\sqrt{2}} d\theta = \frac{2\sqrt{2}}{3}\int_{\theta=\pi/4}^{\pi/2}(\cos\theta+\sin\theta)\,d\theta$$

$$= \frac{2\sqrt{2}}{3}\left[\sin\theta - \cos\theta\right]_{\pi/4}^{\pi/2} = \frac{2\sqrt{2}}{3}\left[(1-0)-\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)\right] = \frac{2\sqrt{2}}{3}$$ **Ans.**

Example 51. *Show that the volume of the wedge intercepted between the cylinder $x^2 + y^2 = 2ax$ and planes $z = mx$, $z = nx$ is $\pi(m - n)a^3$.*

Solution. The equation of the cylinder is $x^2 + y^2 = 2ax$



we convert the cartesian coordinates into cylindrical coordinates.

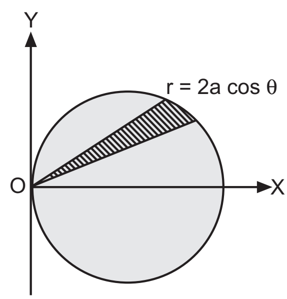
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2 + y^2 = 2ax \quad \Rightarrow \quad r^2 = 2ar\cos\theta$$
$$\Rightarrow \quad r = 2a\cos\theta$$

r varies from 0 to $2a\cos\theta$

θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

and z varies from $z = nx$ $(z = nr\cos\theta)$ to $z = mx$ $(z = mr\cos\theta)$

$$V = 2\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta}\int_{z=nr\cos\theta}^{mr\cos\theta} dz\,dr\,d\theta$$



$$= 2\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r\left[z\right]_{nr\cos\theta}^{mr\cos\theta} dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r\,.(m-n)\,r\cos\theta\,dr\,d\theta$$

$$= 2(m-n)\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r^2\cos\theta\,dr\,d\theta$$

$$= 2(m-n)\int_{\theta=0}^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2a\cos\theta}\cos\theta\,d\theta$$

$$= 2(m-n)\int_{\theta=0}^{\pi/2}\frac{8a^3}{3}\cos^3\theta\cos\theta\,d\theta$$

$$= \frac{16(m-n)}{3}a^3\int_{\theta=0}^{\pi/2}\cos^4\theta\,d\theta = \frac{16(m-n)}{3}.a^3.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = (m-n)\pi a^3$$ **Ans.**

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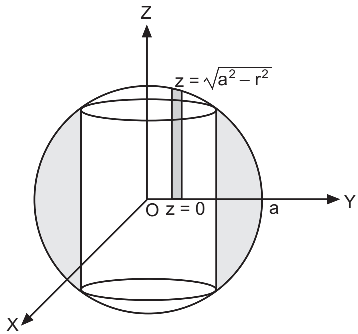
Example 52. *A cylindrical hole of radius b is bored through a sphere of radius a. Find the volume of the remaining solid.*



Solution. Let the equation of the sphere be

$$x^2 + y^2 + z^2 = a^2$$

Now, we will solve this problem using cylindrical coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Limits of z are 0 and $\sqrt{a^2 - (x^2 + y^2)}$ *i.e.*, $\sqrt{a^2 - r^2}$

Limits of r are a and b.

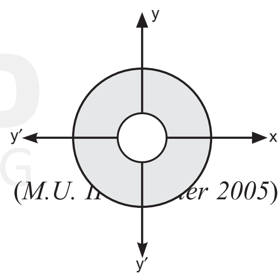
and the limits of θ are 0 and $\dfrac{\pi}{2}$

$$V = 8\int_{\theta=0}^{\pi/2}\int_{r=b}^{a}\int_{z=0}^{\sqrt{a^2-r^2}} r\,dr\,d\theta\,dz$$

$$= 8\int_{\theta=0}^{\pi/2}\int_{r=b}^{a}[z]_0^{\sqrt{a^2-r^2}}\,r\,dr\,d\theta = 8\int_{\theta=0}^{\pi/2}\int_{r=b}^{a}(a^2-r^2)^{1/2}\,.\,r\,dr\,d\theta$$

$$= 8\int_{\theta=0}^{\pi/2}\left[\frac{(a^2-r^2)^{3/2}}{3/2}\cdot\left(-\frac{1}{2}\right)\right]_b^a d\theta = -\frac{8}{3}\int_0^{\pi/2} -(a^2-b^2)^{\frac{3}{2}}\,d\theta$$

$$= \frac{8}{3}(a^2-b^2)^{\frac{3}{2}}[\theta]_0^{\pi/2} = \frac{4\pi}{3}(a^2-b^2)^{\frac{3}{2}} \qquad \textbf{Ans.}$$

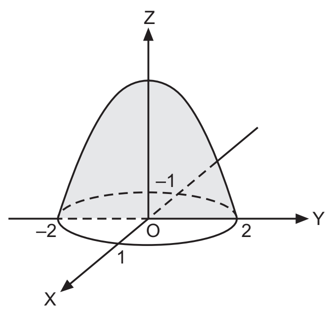


Example 53. *Find the volume cut off from the paraboloid*

$$x^2 + \frac{y^2}{4} + z = 1 \text{ by the plane } z = 0.$$ (*M.U. II ier 2005*)

Solution. We have

$$x^2 + \frac{y^2}{4} + z = 1 \qquad \text{(Paraboloid)} \qquad ...(1)$$
$$z = 0 \qquad (x\text{-}y \text{ plane}) \qquad ...(2)$$



z varies from 0 to $1 - x^2 - \dfrac{y^2}{4}$

y varies from $-2\sqrt{1-x^2}$ to $2\sqrt{1-x^2}$

x varies from –1 to 1.

$$V = \iiint dx\,dy\,dz = \int_{-1}^{1}dx\int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}}dy\int_0^{\left(1-x^2-\frac{y^2}{4}\right)}dz$$

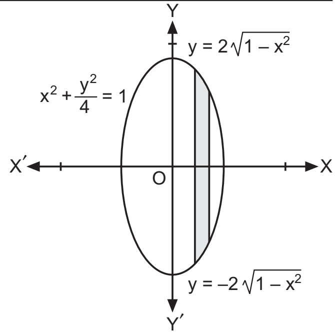
$$= \int_{-1}^{1}\int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}}\left(1-x^2-\frac{y^2}{4}\right)dx\,dy$$

$$= 4\int_0^1\int_0^{2\sqrt{1-x^2}}\left(1-x^2-\frac{y^2}{4}\right)dx\,dy$$

$$= 4\int_0^1 \left[(1-x^2)\,y - \frac{y^3}{12}\right]_0^{2\sqrt{1-x^2}} .\,dx$$

$$= 4\int_0^1 \left[(1-x^2).\,2\sqrt{1-x^2} - \frac{8}{12}(1-x^2)^{3/2}\right] d\,x$$

$$= 4\int_0^1 \left[2(1-x^2)^{3/2} - \frac{2}{3}(1-x^2)^{3/2}\right] d\,x$$



On putting $x = \sin\theta$, we get

$$V = 4\int_0^1 \frac{4}{3}(1-x^2)^{3/2}\,dx = \frac{16}{3}\int_0^{\pi/2}(1-\sin^2\theta)^{3/2}\cos\theta\,d\theta$$

$$= \frac{16}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta = \frac{16}{3}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \pi$$ **Ans.**

Example 54. *Find the volume enclosed between the cylinders $x^2 + y^2 = a\,x$, and $z^2 = a\,x$.*

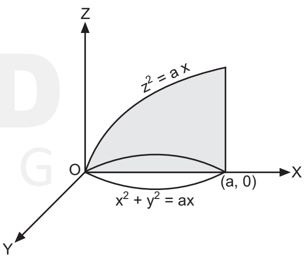
Solution. Here, we have $x^2 + y^2 = ax$ ...(1)

$z^2 = ax$ ...(2)

$$V = \iiint dx\,dy\,dz = \int_0^a dx\int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} dy\int_{-\sqrt{ax}}^{\sqrt{ax}} dz = 2\int_0^a dx\int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} dy\int_0^{\sqrt{ax}} dz$$

$$= 2\int_0^a dx\int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} dy\,(z)_0^{\sqrt{ax}} = 2\int_0^a dx\int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} dy\,\sqrt{ax} = 2\int_0^a \sqrt{ax}\,dx\,[y]_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}}$$

$$= 2\int_0^a \sqrt{ax}\,dx\left(2\sqrt{ax-x^2}\right) = 4\sqrt{a}\int_0^a x\sqrt{a-x}\,dx$$



Putting $x = a\sin^2\theta$ so that $dx = 2a\sin\theta\cos\theta\,d\theta$, we get

$$V = 4\sqrt{a}\int_0^{\pi/2} a\sin^2\theta\sqrt{a - a\sin^2\theta}\,.\,2a\sin\theta\cos\theta\,d\theta$$

$$= 8a^3\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta$$

$$= 8a^3\,\frac{\Gamma(2)\,\Gamma\left(\frac{3}{2}\right)}{2\,\Gamma\left(\frac{7}{2}\right)} = 4a^3\,\frac{\Gamma\left(\frac{3}{2}\right)}{\frac{5}{2}.\frac{3}{2}\,\Gamma\left(\frac{3}{2}\right)} = \frac{16a^3}{15}$$ **Ans.**

EXERCISE 33.10

1. Find the volume bounded by the coordinate planes and the plane. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$
   (*A.M.I.E.T.E. Dec. 2017*)
2. Find the volume bounded by the cylinders $y^2 = x$ and $x^2 = y$ between the planes $z = 0$ and $x + y + z = 2$.
3. Find the volume bounded by the co-ordinate planes and the plane.
   $$l\,x + m\,y + n\,z = 1$$
4. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ by triple integration. **(*AMIETE June 2009*)**

5. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

6. Find the volume bounded by the cylinder $x^2 + y^2 = a^2$ and the planes $y + z = 2a$ and $z = 0$. **(*M.U. II Semester 2000, 02, 06*)**

7. Find the volume bounded by the cylinder $x^2 + y^2 = a^2$ and the planes $z = 0$ and $y + z = b$.

8. Find the volume of the region bounded by $z = x^2 + y^2$, $z = 0$, $x = -a$, $x = a$ and $y = -a$, $y = a$.

9. Find the volume enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $x + z = 5$ and $z = 0$.

10. Compute the volume of the solid bounded by $x^2 + y^2 = z$, $z = 2x$.

11. Find the volume cut from the paraboloid $4z = x^2 + y^2$ by plane $z = 4$. **(*U.P. I Semester Dec. 2005*)**

12. By using triple integration find the volume cut off from the sphere $x^2 + y^2 + z^2 = 16$ by the plane $z = 0$ and the cylinder $x^2 + y^2 = 4x$.

13. The sphere $x^2 + y^2 + z^2 = a^2$ is pierced by the cylinder $x^2 + y^2 = a^2(x^2 - y^2)$.
Prove that the volume of the sphere that lies inside the cylinder is $\frac{8}{3}\left[\frac{\pi}{4}+\frac{5}{3}-\frac{4\sqrt{2}}{3}\right]a^3$.

14. Find the volume of the solid bounded by the surfaces $z = 0$, $3z = x^2 + y^2$ and $x^2 + y^2 = 9$. **(*A.M.I.E.T.E., Summer 2005*)**

15. Obtain the volume bounded by the surface $z = c\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right)$ and a quadrant of the elliptic cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $z > 0$ and where $a, b > 0$. **(*A.M.I.E.T.E. Dec. 2005*)**

16. Find the volume of the paraboloid $x^2 + y^2 = 4z$ cut off by the plane $z = 4$.

17. Find the volume bounded by the cone $z^2 = x^2 + y^2$ and the paraboloid $z = x^2 + y^2$.

18. Find the volume enclosed by the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$.

19. Find the volume of the solid bounded by the plane $z = 0$, the paraboloid $z = x^2 + y^2 + 2$ and the cylinder $x^2 + y^2 = 4$.

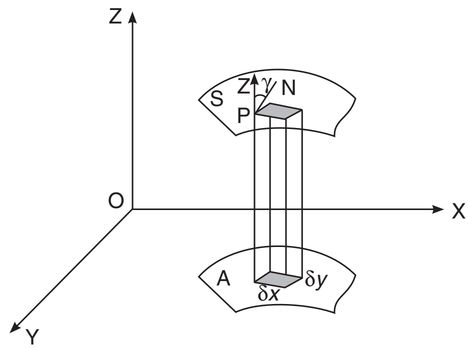
20. The triple integral $\iiint dx\,dy\,dz$ gives
(*a*) Volume of region (*b*) Surface area of region T
(*c*) Area of region *T* (*d*) Density of region *T*. **(*A.M.I.E.T.E. Dec. 2006*)**

ANSWERS

1. $\frac{abc}{6}$	**2.** $\frac{11}{30}$	**3.** $\frac{1}{6lmn}$	**4.** $\frac{4}{3}\pi a^3$
5. $\frac{4\pi abc}{3}$	**6.** $2\pi a^3$	**7.** $\pi a^2 b$	**8.** $\frac{8}{3}a^4$
9. $45\pi - 36$	**10.** 2π	**11.** 32π	**12.** $\frac{64}{9}(3\pi - 4)$
14. $\frac{27\pi}{2}$	**15.** πabc	**16.** 32π	**17.** $\frac{\pi}{6}$
18. $\frac{128a^3}{15}$	**19.** 16π	**20.** (*a*)	

33.16 SURFACE AREA

Let $z = f(x,y)$ be the surface S. Let its projection on the x-y plane be the region A. Consider an element $\delta x.\ \delta y$ in the region A. Erect a cylinder on the element $\delta x.\ \delta y$ having its generator parallel to OZ and meeting the surface S in an element of area δs.



$\therefore \qquad \delta x\ \delta y = \delta s \cos\gamma,$

Where γ is the angle between the xy-plane and the tangent plane to S at P, *i.e.,* it is the angle between the Z-axis and the normal to S at P.

The direction cosines of the normal to the surface $F(x, y, z) = 0$ are proportional to

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$$

$\therefore$ The direction of the normal to S $[F = f(x, y) - z]$ are proportional to $-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1$ and those of the Z-axis are 0, 0 , 1.

$$\text{Direction cosines} = \frac{-\frac{\partial z}{\partial x}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}, \frac{-\frac{\partial z}{\partial y}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}, \frac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}},$$

Hence
$$\cos\gamma = \frac{1}{\sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}} \quad (\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2)$$

$$\delta S = \frac{\delta x\ \delta y}{\cos\gamma} = \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\ \delta x\ \delta y\ ; \quad S = \iint_A \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\ dx\ dy$$

Example 55. *Find the surface area of the cylinder* $x^2 + z^2 = 4$ *inside the cylinder* $x^2 + y^2 = 4$.

Solution. $x^2 + y^2 = 4$

$x^2 + z^2 = 4$

$$2x + 2z\frac{\partial z}{\partial x} = 0 \text{ or } \frac{\partial z}{\partial x} = -\frac{x}{z}, \ \frac{\partial z}{\partial y} = 0$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = \frac{x^2}{z^2} + 1 = \frac{x^2 + z^2}{z^2} = \frac{4}{4 - x^2}$$

$$\text{Hence, the required surface area} = 8\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\ dx\ dy$$

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$$= 8\int_0^2\int_0^{\sqrt{4-x^2}} \frac{2}{\sqrt{4-x^2}}\,dx\,dy = 16\int_0^2 \frac{1}{\sqrt{4-x^2}}[y]_0^{\sqrt{4-x^2}}\,dx = 16\int_0^2 \frac{1}{\sqrt{4-x^2}}[\sqrt{4-x^2}]\,dx$$

$$= 16\int_0^2 dx = 16\,(x)_0^2 = 32$$ **Ans.**

Example 56. *Find the surface area of the sphere $x^2 + y^2 + z^2 = 9$ lying inside the cylinder $x^2 + y^2 = 3y$.*

Solution.
$$x^2 + y^2 + z^2 = 9$$
$$2x + 2z\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial x} = -\frac{x}{z}$$
$$2y + 2z\frac{\partial z}{\partial y} = 0, \quad \frac{\partial z}{\partial y} = -\frac{y}{z}$$

$$\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right] = \frac{x^2}{z^2} + \frac{y^2}{z^2} + 1 = \frac{x^2+y^2+z^2}{z^2} = \frac{9}{9-x^2-y^2} = \frac{9}{9-r^2} \quad \begin{bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{bmatrix}$$

$$x^2 + y^2 = 3y \quad \text{or } r^2 = 3\,r\sin\theta \quad \text{or} \quad r = 3\sin\theta.$$

Hence, the required surface area

$$= \iint \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\,dx\,dy = 4\int_0^{\pi/2}\int_0^{3\sin\theta} \frac{3}{\sqrt{9-r^2}}\,r\,d\theta\,dr = 12\int_0^{\pi/2} d\theta \int_0^{3\sin\theta} \frac{r\,dr}{\sqrt{9-r^2}}$$

$$= 12\int_0^{\pi/2} d\theta\,[-\sqrt{9-r^2}]_0^{3\sin\theta} = 12\int_0^{\pi/2} [-\sqrt{9-9\sin^2\theta} + 3]\,d\theta$$

$$= 36\int_0^{\pi/2} (-\cos\theta + 1)\,d\theta = 36\,(-\sin\theta + \theta)_0^{\pi/2} = 36\left(-1 + \frac{\pi}{2}\right) = 18\,(\pi - 2)$$ **Ans.**

Example 57. *Find the surface area of the section of the cylinder $x^2 + y^2 = a^2$ made by the plane $x + y + z = a$.*

Solution.
$$x^2 + y^2 = a^2 \qquad \text{...(1)}$$
$$x + y + z = a \qquad \text{...(2)}$$

The projection of the surface area on xy-plane is a circle

$$x^2 + y^2 = a^2$$

$$1 + \frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial z}{\partial x} = -1$$
$$1 + \frac{\partial z}{\partial y} = 0 \quad \text{or} \quad \frac{\partial z}{\partial y} = -1$$

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{(-1)^2 + (-1)^2 + 1} = \sqrt{3}$$

Hence the required surface area

$$= 4\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dx\,dy = 4\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{3}\,dx \cdot dy$$

$$= 4\sqrt{3}\int_0^a [y]_0^{\sqrt{a^2-x^2}}\,dx = 4\sqrt{3}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= 4\sqrt{3}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a = 4\sqrt{3}\left[0 + \frac{a^2}{2}\frac{\pi}{2}\right] = 4\sqrt{3}\left(\frac{a^2\pi}{4}\right) = \sqrt{3}\,\pi\, a^2 \quad \textbf{Ans.}$$

Example 58. *Find the area of that part of the surface of the paraboloid $y^2 + z^2 = 2\,ax$, which lies between the cylinder, $y^2 = ax$ and the plane $x = a$.*

Solution.

$$y^2 + z^2 = 2\,ax \qquad ...(1)$$

$$y^2 = ax \qquad ...(2)$$

$$x = a \qquad ...(3)$$

Differentiating (1), we get

$$2\,z\frac{\partial z}{\partial x} = 2\,a, \quad \frac{\partial z}{\partial x} = \frac{a}{z}$$

$$2y + 2\,z\frac{\partial z}{\partial y} = 0, \quad \frac{\partial z}{\partial y} = -\frac{y}{z}$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = \frac{a^2}{z^2} + \frac{y^2}{z^2} + 1 = \frac{a^2 + y^2}{z^2} + 1 \qquad \begin{bmatrix} y^2 + z^2 = 2\,ax \\ z^2 = 2\,ax - y^2 \end{bmatrix}$$

$$= \frac{a^2 + y^2}{2\,a\,x - y^2} + 1 = \frac{a^2 + y^2 + 2\,a\,x - y^2}{2\,a\,x - y^2} = \frac{a^2 + 2\,a\,x}{2\,a\,x - y^2}$$

$$S = \int_0^a \int_{-\sqrt{ax}}^{\sqrt{ax}} \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dx\,dy = \int_0^a \int_{-\sqrt{ax}}^{\sqrt{ax}} \sqrt{\frac{a^2 + 2\,ax}{2\,ax - y^2}}\; dx\,dy \qquad \begin{bmatrix} y^2 = ax \\ y = \pm\sqrt{ax} \end{bmatrix}$$

$$= \sqrt{a}\int_0^a \int_{-\sqrt{ax}}^{\sqrt{ax}} \sqrt{\frac{a + 2\,x}{2\,ax - y^2}}\; dx\,dy = \sqrt{a}\int_0^a \sqrt{a + 2\,x}\; dx \int_{-\sqrt{ax}}^{\sqrt{ax}} \frac{1}{\sqrt{2\,ax - y^2}}\,dy$$

$$= \sqrt{a}\int_0^a \sqrt{a + 2\,x}\; dx\left[\sin^{-1}\frac{y}{\sqrt{2ax}}\right]_{-\sqrt{ax}}^{\sqrt{ax}} = \sqrt{a}\int_0^a \sqrt{a + 2\,x}\; dx\left[\sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right]$$

$$= \sqrt{a}\int_0^a \sqrt{a + 2\,x}\; dx\left[\frac{\pi}{4} + \left(\frac{\pi}{4}\right)\right] = \sqrt{a}\,\frac{\pi}{2}\int_0^a \sqrt{a + 2\,x}\; dx = \frac{\pi}{2}\cdot\frac{\sqrt{a}}{2}\cdot\frac{2}{3}\left[(a + 2x)^{3/2}\right]_0^a$$

$$= \frac{\pi\sqrt{a}}{6}\left[(3a)^{3/2} - a^{3/2}\right] = \frac{\pi\, a^2}{6}\left[3\sqrt{3} - 1\right] \qquad \textbf{Ans.}$$

EXERCISE 33.11

1. Find the surface area of sphere $x^2 + y^2 + z^2 = 16$.
2. Find the surface area of the portion of the cylinder $x^2 + y^2 = 4\,y$ lying inside the sphere $x^2 + y^2 + z^2 = 16$.
3. Show that the area of surfaces $cz = xy$ intercepted by the cylinder $x^2 + y^2 = b^2$ is $\iint_A \frac{\sqrt{c^2 + x^2 + y^2}}{c}\,dx\,dy$, where A is the area of the circle $x^2 + y^2 = b^2$, $z = 0$
4. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + \mathbf{y}^2 = ax$.
5. Find the area of the surface of the cone $z^2 = 3\,(x^2 + y^2)$ cut out by the paraboloid $z = x^2 + y^2$ using surface integral.

ANSWERS

1. 64π	**2.** 64.	**3.** $\frac{2}{3}\frac{\pi}{c}\left[(c^2+b^2)^{\frac{1}{2}}-c^2\right]$
4. $2(\pi-2)a^2$	**5.** 6π	

33.17 CALCULATION OF MASS

We have,

Volume $= \iiint_V dx\, dy\, dz$ Density $= \rho = f(x, y, z)$

[Density = Mass per unit volume]   Mass = Volume × Density

Volume $= \iiint_V dx\, dy\, dz$ $\boxed{Mass = \iiint_V f(x, y, z)\, dx\, dy\, dz}$

Example 59. Find the mass of a plate which is formed by the co-ordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$ the density is given by $\rho = k\,x\,y\,z$. **(*U.P. I Semester Dec.* 2003)**

Solution. The plate is bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

$$\text{Mass} = \iiint dx\, dy\, dz\, \rho = \int_0^c \int_0^{b\left(1-\frac{z}{c}\right)} \int_0^{a\left(1-\frac{y}{b}-\frac{z}{c}\right)} dx\, dy\, dz\,(k\,xyz)$$

$$= k\int_0^c z\,dz \int_0^{b\left(1-\frac{z}{c}\right)} y\,dy \int_0^{a\left(1-\frac{y}{b}-\frac{z}{c}\right)} x\,dx = k\int_0^c z\,dz \int_0^{b\left(1-\frac{z}{c}\right)} y\,dy \left(\frac{x^2}{2}\right)_0^{a\left(1-\frac{y}{b}-\frac{z}{c}\right)}$$

$$= k\int_0^c z\,dz \int_0^{b\left(1-\frac{z}{c}\right)} y\,dy\,\frac{a^2}{2}\left(1-\frac{y}{b}-\frac{z}{c}\right)^2 = \frac{k\,a^2}{2}\int_0^c z\,dz \int_0^{b\left(1-\frac{z}{c}\right)} y\left[\left(1-\frac{z}{c}\right)-\frac{y}{b}\right]^2 dy$$

$$= \frac{k\,a^2}{2}\int_0^c z\,dz \int_0^{b\left(1-\frac{z}{c}\right)} \left[y\left(1-\frac{z}{c}\right)^2+\frac{y^3}{b^2}-\frac{2\,y^2}{b}\left(1-\frac{z}{c}\right)\right]dy$$

$$= \frac{k\,a^2}{2}\int_0^c z\,dz \left[\frac{y^2}{2}\left(1-\frac{z}{c}\right)^2+\frac{y^4}{4\,b^2}-\frac{2\,y^3}{3\,b}\left(1-\frac{z}{c}\right)\right]_0^{b\left(1-\frac{z}{c}\right)}$$

$$= \frac{k\,a^2}{2}\int_0^c z\,dz \left[\frac{b^2}{2}\left(1-\frac{z}{c}\right)^4+\frac{b^4}{4\,b^2}\left(1-\frac{z}{c}\right)^4-\frac{2}{3}\cdot\frac{b^3}{b}\left(1-\frac{z}{c}\right)^4\right]$$

$$= \frac{k\,a^2}{2}\int_0^c z\left[\frac{b^2}{2}+\frac{b^2}{4}-\frac{2b^2}{3}\right]\left(1-\frac{z}{c}\right)^4 dz = \frac{k\,a^2}{2}\,\frac{b^2}{12}\int_0^c \left(1-\frac{z}{c}\right)_z^4 dz \quad [\text{Put } z = c\sin^2\theta]$$

$$= \frac{k\,a^2\,b^2}{24}\int_0^{\frac{\pi}{2}} c\sin^2\theta\,(1-\sin^2\theta)^4\,(2\,c\sin\theta\cos\theta\,d\theta)$$

$$= \frac{k^2\,a^2\,b^2c^2}{12}\int_0^{\pi/2}\sin^2\theta\,(\cos^8\theta)\sin\theta\cos\theta\,d\theta = \frac{k^2\,a^2\,b^2\,c^2}{12}\int_0^{\pi/2}\sin^3\theta\cos^9\theta\,d\theta$$

$$= \frac{k^2\, a^2\, b^2\, c^2}{12} \frac{\left|\frac{3+1}{2}\right.\left|\frac{9+1}{2}\right.}{2\left|\frac{3+9+2}{2}\right.} = \frac{k\, a^2\, b^2\, c^2}{12} \cdot \frac{|2\,|5}{2\,|7} = \frac{k\, a^2\, b^2\, c^2}{12} \frac{(1)\,(|5)}{2 \times 6 \times 5\,|5} = \frac{k\, a^2\, b^2\, c^2}{720}$$ **Ans.**

33.18 CENTRE OF GRAVITY

$$\bar{x} = \frac{\iiint x\, \rho\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz},\ \bar{y} = \frac{\iiint y\, \rho\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz},\ \bar{z} = \frac{\iiint z\, \rho\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz}$$

Example 60. *Find the co-ordinates of the centre of gravity of the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$, density being given = k xyz.*

Solution. $$\bar{x} = \frac{\iiint_V x\, \rho\, dx\, dy\, dz}{\iiint_V \rho\, dx\, dy\, dz} = \frac{\iiint z\, \rho\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz} = \frac{\iiint_V x^2 yz\, dx\, dy\, dz}{\iiint_V xyz\, dx\, dy\, dz}$$

Converting into polar co-ordinates, $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$,

$dx\, dy\, dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$

$$\bar{x} = \frac{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a (r\sin\theta\cos\phi)^2\,(r\sin\theta\sin\phi)\,(r\cos\theta)\,(r^2\sin\theta\, dr\, d\theta\, d\phi)}{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a (r\sin\theta\cos\phi)\,(r\sin\theta\sin\phi)\,(r\cos\theta)\,(r^2\sin\theta\, dr\, d\theta\, d\phi)}$$

$$= \frac{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a r^6 \sin^4\theta\cos\theta\sin\phi\cos^2\phi\, dr\, d\theta\, d\phi}{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a r^5 \sin^3\theta\cos\theta\sin\phi\cos\phi\, dr\, d\theta\, d\phi}$$

$$= \frac{\int_0^{\pi/2}\sin\phi\cos^2\phi\, d\phi \int_0^{\pi/2}\sin^4\theta\cos\theta\, d\theta \int_0^a r^6\, dr}{\int_0^{\pi/2}\sin\phi\cos\phi\, d\phi \int_0^{\pi/2}\sin^3\theta\cos\theta\, d\theta \int_0^a r^5\, dr}$$

$$= \frac{\left[-\frac{\cos^3\phi}{3}\right]_0^{\pi/2}\left[\frac{\sin^5\theta}{5}\right]_0^{\pi/2}\left[\frac{r^7}{7}\right]_0^a}{-\left[\frac{\cos^2\phi}{2}\right]_0^{\pi/2}\left[\frac{\sin^4\theta}{4}\right]_0^{\pi/2}\left[\frac{r^6}{6}\right]_0^a} = \frac{\left(\frac{1}{3}\right)\left(\frac{1}{5}\right)\left(\frac{a^7}{7}\right)}{\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{a^6}{6}\right)} = \frac{16\,a}{35}$$

Similarly, $\bar{y} = \bar{z} = \frac{16\,a}{35}$; Hence, C.G. is $\left(\frac{16\,a}{35}, \frac{16\,a}{35}, \frac{16\,a}{35}\right)$ **Ans.**

33.19 MOMENT OF INERTIA OF A SOLID

Let the mass of an element of a solid of volume V be $\rho\ \delta x\, \delta y\, \delta z$.

Perpendicular distance of this element from the x-axis = $\sqrt{y^2 + z^2}$

M.I. of this element about the x-axis = $\rho\, \delta x\, \delta y\, \delta z \sqrt{y^2 + z^2}$

M.I. of the solid about x-axis = $\iiint_V \rho\,(y^2 + z^2)\, dx\, dy\, dz$

M.I. of the solid about y-axis = $\iiint_V \rho\,(x^2 + z^2)\, dx\, dy\, dz$

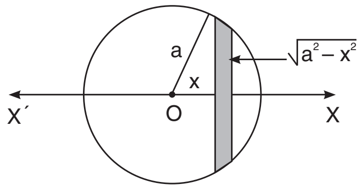
M.I. of the solid about *z*-axis = $\iiint_V \rho\,(x^2 + y^2)\,dx\,dy\,dz$

The Perpendicular Axes Theorem

If I_{ox} and I_{oy} be the moments of inertia of a lamina about *x*-axis and *y*-axis respectively and I_{oz} be the moment of inertia of the lamina about an axis perpendicular to the lamina and passing through the point of intersection of the axes *OX* and *OY*.

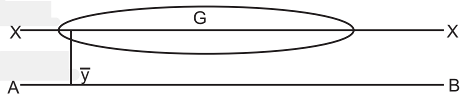
$$I_{OZ} = I_{OX} + I_{OY}$$

The Parallel Axes Theorem



M.I. of a lamina about an axis in the plane of the lamina equals the sum of the moment of inertia about a parallel centroidal axis in the plane of lamina together with the product of the mass of the lamin *a* and square of the distance between the two axes.

$$I_{AB} = I_{XX} + My^2$$



Example 61. *Find M.I. of a sphere about diameter.*

Solution. Let a circular disc of δx thickness be perpendicular to the given diameter *XX′* at a distance *x* from it.

$$\text{The radius of the disc} = \sqrt{a^2 - x^2}$$

$$\text{Mass of the disc} = \rho\,\pi\,(a^2 - x^2)$$

Moment of inertia of the disc about a diameter perpendicular on it

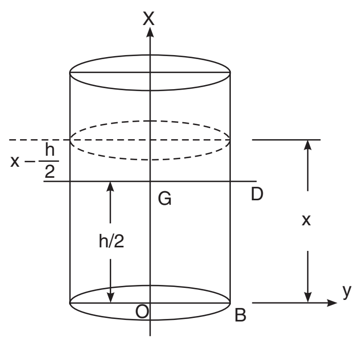
$$= \frac{1}{2} MR^2 = \frac{1}{2}[\rho\,\pi\,(a^2 - x^2)]\,(a^2 - x^2) = \frac{1}{2}\rho\,\pi\,(a^2 - x^2)^2$$

$$\text{M.I. of the sphere} = \int_{-a}^{a} \frac{1}{2}\rho\,\pi\,(a^2 - x^2)^2\,dx = 2\left(\frac{1}{2}\rho\,\pi\right)\int_0^a [a^4 - 2\,a^2\,x^2 + x^4]\,dx$$

$$= \rho\,\pi\left[a^4 x - \frac{2\,a^2\,x^3}{3} + \frac{x^5}{5}\right]_0^a = \rho\,\pi\left[a^5 - \frac{2\,a^5}{3} + \frac{a^5}{5}\right]$$

$$= \frac{8}{15}\pi\,\rho\,a^5 = \frac{2}{5}\left(\frac{4\,\pi}{3}a^3\,\rho\right)a^2 = \frac{2}{5}M\,a^2$$ **Ans.**

Example 62. *The mass of a solid right circular cylinder of radius a and height h* is *M. Find the moment of inertia of the cylinder about* (*i*) *its axis* (*ii*) *a line through its centre of gravity perpendicular to its axis* (*iii*) *any diameter through its base.*



Solution. *To find M.I. about OX. Consider a disc at a distance x from O at the base.*

M.I. *of the about OX*, $= \dfrac{(\pi\,a^2\,\rho\,dx)\,a^2}{2} = \dfrac{\pi\,\rho\,a^4\,dx}{2}$

(*i*) M.I. of the cylinder about OX

$$\int_0^h \frac{\pi\,\rho\,a^4 dx}{2} = \frac{\pi\,\rho\,a^4}{2}(x)_0^h = \frac{\pi\,\rho\,a^4\,h}{2} = (\pi\,a^2\,h)\,\rho\cdot\frac{a^2}{2} = \frac{M\,a^2}{2}$$

(*ii*) *M.I.* of the disc about a line through *C.G.* and perpendicular to *OX*.

$$I_{OX} + I_{OY} = I_{OZ}$$
$$I_{OX} + I_{OX} = I_{OZ}$$
$$I_{OX} = \frac{1}{2} I_{OZ}$$

M.I. of the disc about a line through

$$C.G. = \frac{1}{2}\left(\frac{M\,a^2}{2}\right) = \frac{M\,a^2}{4}$$

M.I. of the disc about the diameter $= \left(\dfrac{\pi\,a^2\,\rho\,dx}{4}\right) a^2$

M.I. of the disc about line $GD = \dfrac{\pi\,a^4\,\rho\,dx}{4} + (\pi\,a^2\,\rho\,dx)\left(x - \dfrac{h}{2}\right)^2$

Hence, *M.I.* of cylinder about $GD = \displaystyle\int_0^h \frac{\pi\,a^4\,\rho}{4}\,dx + \int_0^h (\pi\,a^2\,\rho\,dx)\left(x - \frac{h}{2}\right)^2$

$$= \frac{\pi\,a^2\,\rho}{4}(x)_0^h + \left[\frac{\pi\,a^2\,\rho}{4}\left(x - \frac{h}{2}\right)^3\right]_0^h = \frac{\pi\,a^4\,\rho\,h}{4} + \left[\frac{\pi\,a^2\,\rho}{3}\left(\frac{h}{2}\right)^3 + \frac{\pi\,a^2\,\rho}{3}\left(\frac{h}{2}\right)^3\right]$$

$$= \frac{\pi\,a^4\,\rho\,h}{4} + \frac{\pi\,a^2\,\rho\,h^3}{12} = \frac{M\,a^2}{4} + \frac{M\,h^2}{12}$$

(*iii*) *M.I.* of cylinder about line *OB* (through) base

$$I_{OB} = I_{GD} + M\left(\frac{h}{2}\right)^2 = \frac{M\,a^2}{4} + \frac{M\,h^2}{12} + \frac{M\,h^2}{4} = \frac{M\,a^2}{4} + \frac{M\,h^2}{3}$$ **Ans.**

Example 63. *Find the moment of inertia and radius of gyration about z-axis of the region in the first octant bounded by* $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$.

Solution. Let *r* be the density. M.I. of tetrahedron about *z*-axis

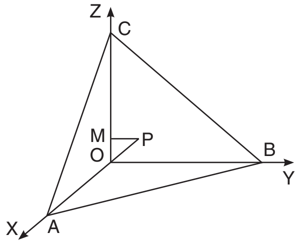
$$= \iiint (\rho\,dx\,dy\,dz)\,(x^2 + y^2) = \rho\int_0^a dx \int_0^{b\left(1-\frac{x}{a}\right)} (x^2 + y^2)\,dy \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz$$

$$= \rho\int_0^a dx \int_0^{b\left(1-\frac{x}{a}\right)} (x^2 + y^2)\,dy\,(z)_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} = \rho\int_0^a dx \int_0^{b\left(1-\frac{x}{a}\right)} (x^2 + y^2)\,dy\;c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$$

$$= \rho c\int_0^a dx \int_0^{b\left(1-\frac{x}{a}\right)} \left[x^2\left(1 - \frac{x}{a}\right) - \frac{x^2 y}{b} + y^2\left(1 - \frac{x}{a}\right) - \frac{y^3}{b}\right] dy$$

$$= \rho c\int_0^a dx \left[x^2\left(1 - \frac{x}{a}\right) y - \frac{x^2 y^2}{2\,b} + \frac{y^3}{3}\left(1 - \frac{x}{a}\right) - \frac{y^4}{4\,b}\right]_0^{b\left(1-\frac{x}{a}\right)}$$

$$= \rho c\int_0^a dx \left[x^2\left(1 - \frac{x}{a}\right) b\left(1 - \frac{x}{a}\right) - \frac{x^2}{2\,b} b^2\left(1 - \frac{x}{a}\right)^2\right.$$



$$+\frac{b^3}{3}\left(1-\frac{x}{a}\right)^3\left(1-\frac{x}{a}\right)-\frac{b^4}{4\,b}\left(1-\frac{x}{a}\right)^4\Bigg]$$

$$= b\rho c\int_0^a\left[x^2\left(1-\frac{x}{a}\right)^2-\frac{x^2}{2}\left(1-\frac{x}{a}\right)^2-\frac{b^2}{3}\left(1-\frac{x}{a}\right)^4-\frac{b^2}{4}\left(1-\frac{x}{a}\right)^4\right]dx$$

$$= \rho bc\int_0^a\left[\frac{x^2}{2}\left(1-\frac{x}{a}\right)^2+\frac{b^2}{12}\left(1-\frac{x}{a}\right)^4\right]dx$$

$$= \rho bc\int_0^a\left[\frac{1}{2}\left(x^2-\frac{2\,x^3}{a}+\frac{x^4}{a^2}\right)+\frac{b^2}{12}\left(1-\frac{4\,x}{a}+\frac{6\,x^2}{a^2}-\frac{4\,x^3}{a^3}+\frac{x^4}{a^4}\right)\right]dx$$

$$= \rho bc\left[\frac{1}{2}\left(\frac{x^3}{3}-\frac{x^4}{2\,a}+\frac{x^5}{5\,a^2}\right)+\frac{b^2}{12}\left(x-\frac{2\,x^2}{a}+\frac{6\,x^3}{3a^2}-\frac{4\,x^3}{4a^3}+\frac{x^5}{5a^4}\right)\right]_0^a$$

$$= \rho bc\left[\frac{1}{2}\left(\frac{a^3}{3}-\frac{a^3}{2}+\frac{a^3}{5}\right)+\frac{b^2}{12}\left(a-2a+2a-a+\frac{a}{5}\right)\right]$$

$$= \rho bc\left[\frac{a^3}{60}+\frac{ab^2}{60}\right] = \rho\,\frac{abc}{60}\,(a^2+b^2)$$

Radius of gyration $= \sqrt{\dfrac{M.I.}{Mass}} = \dfrac{\sqrt{\dfrac{\rho abc}{60}\,(a^2+b^2)}}{\dfrac{\rho abc}{6}} = \sqrt{\dfrac{1}{10}\,(a^2+b^2)}$ **Ans.**

33.20 CENTRE OF PRESSURE

The centre of pressure of a plane area immersed in a fluid is the point at which the resultant force acts on the area.

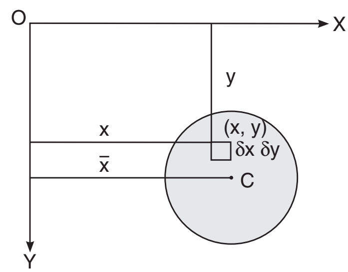
Consider a plane area A immersed vertically in a homogeneous liquid. Let x-axis be the line of intersection of the plane with the free surface. Any line in this plane and perpendicular to x-axis is the *y-axis.*

Let P be the pressure at the point (x, y). Then the pressure on elementary area $\delta x\ \delta y$ is $P\ \delta x\ \delta y$. Let $(\bar{x}, \bar{y})$ be the centre of pressure. Taking moment about y-axis.

$$\bar{x}\cdot\iint_A P\,dx\,dy = \iint_A Px\,dx\,dy$$

$$\bar{x} = \frac{\iint_A Px\,dx\,dy}{\iint_A P\,dx\,dy}$$

Similarly,
$$\bar{y} = \frac{\iint_A Py\,dx\,dy}{\iint_A P\,dx\,dy}$$



Example 64. *A uniform semi-circular lamina is immersed in a fluid with its plane vertical and its bounding diameter on the free surface. If the density at any point of the fluid varies as the depth of the point below the free surface, find the position of the centre of*

pressure of the lamina.

Solution. Let the semi-circular lamina be

$$x^2 + y^2 = a^2$$

By symmetry its centre of pressure lies on OY. Let ky be the density of the fluid.

$$\bar{y} = \frac{\iint_A Py\,dx\,dy}{\iint_A P\,dx\,dy} = \frac{\iint_A (\rho y)\,y\,dx\,dy}{\iint_A (\rho y)\,dx\,dy} \qquad (\because \rho = ky)$$

$$= \frac{\iint_A (ky\,.\,y)\,y\,dx\,dy}{\iint_A (ky\,.\,y)\,dx\,dy} = \frac{\iint_A y^3\,dx\,dy}{\iint_A y^2\,dx\,dy} = \frac{\int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} y^3\,dy}{\int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} y^2\,dy} = \frac{\int_{-a}^{a} dx \left[\frac{y^4}{4}\right]_0^{\sqrt{a^2-x^2}}}{\int_{-a}^{a} dx \left[\frac{y^3}{3}\right]_0^{\sqrt{a^2-x^2}}}$$

$$= \frac{3}{4}\,\frac{\int_{-a}^{a} dx\,(a^2-x^2)^2}{\int_{-a}^{a} dx\,(a^2-x^2)^{3/2}} = \frac{3}{4}\,\frac{\int_{-\pi/2}^{\pi/2} (a\cos\theta\,d\theta)\,(a^2-a^2\sin^2\theta)^2}{\int_{-\pi/2}^{\pi/2} (a\cos\theta\,d\theta)\,(a^2-a^2\sin^2\theta)^{3/2}} \qquad (\text{Put } x = a\sin\theta)$$

$$= \frac{3\,a}{4}\,\frac{\int_{-\pi/2}^{\pi/2} \cos^5\theta\,d\theta}{\int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta} = \frac{3\,a}{4}\,\frac{2\int_0^{\pi/2} \cos^5\theta\,d\theta}{2\int_0^{\pi/2} \cos^4\theta\,d\theta} = \frac{3\,a}{4}\,\frac{\frac{4\times 2}{5\times 3}}{\frac{3\times 1}{4\times 2}\,\frac{\pi}{2}} = \frac{32\,a}{15\,\pi} \qquad \textbf{Ans.}$$

EXERCISE 33.12

1. Find the mass of the solid bounded by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and the co-ordinate planes, where the density at any point $P\,(x, y, z)$ is $k\,xyz$.
2. If the density at a point varies as the square of the distance of the point from XOY plane, find the mass of the volume common to the sphere $x^2 + y^2 + z^2 = a^2$ and cylinder $x^2 + y^2 = ax$.
3. Find the mass of the plate in the form of one loop of leminscate $r^2 = a^2 \sin 2\,\theta$, where $\rho = k\,r^2$.
4. Find the mass of the plate which is inside the circle $r = 2a\cos\theta$ and outside the circle $r = a$, if the density varies as the distance from the pole.
5. Find the mass of a lamina in the form of the cardioid $r = a\,(1 + \cos\theta)$ whose density at any point varies as the square of its distance from the initial line.
6. Find the centroid of the region in the first octant bounded by $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.
7. Find the centroid of the region bounded by $z = 4 - x^2 - y^2$ and xy-plane.
8. Find the position of $C.G.$ of the volume intercepted between the parallelepiped $x^2+y^2 = a(a - z)$ and the plane $z = 0$.
9. A solid is cut off the cylinder $x^2 + y^2 = a^2$ by the plane $z = 0$ and that part of the olane $z = mx$ for which z is positive. The density of the solid cut off at any point varies as the height of the point above plane $z = 0$. Find $C.G.$ of the solid.
10. If an area is bounded by two concentric semi-circles with their common bounding diameter in a free surface, prove that the depth of the centre of pressure is $\frac{3\,\pi}{16}\,\frac{(a+b)\,(a^2+b^2)}{a^2+ab+b^2}$

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11. An ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is immersed vertically in a fluid with its major axis horizontal. If its centre be at depth *h,* find the depth of its centre of pressure.

12. A horizontal boiler has a flat bottom and its ends are plane and semi-circular. If it is just full of water, show that the depth of centre of pressure of either end is 0.7 × total depth approximately.

13. A quadrant of a circle of radius *a* is just immersed vertically in a homogeneous liquid with one edge in the surface. Determine the co-ordinates of the centre of pressure.

14. Find the product of inertia of an equilateral triangle about two perpendicular axes in its plane at a vertex, one of the axes being along a side.

15. Find the *M.I.* of a right circular cylinder of radius *a* and height *h* about axis if density varies as distance from the axis.

16. Compute the moment of inertia of a right circular cone whose altitude is *h* and base radius *r,* about (*i*) the axis of symmetry (*ii*) the diameter of the base.

17. Find the moment of inertia for the area of the cardioid $r = a\,(1 - \cos\theta)$ relative to the pole.

18. Find the M.I. about the line $\theta = \frac{\pi}{2}$ of the area enclosed by $r = a\,(1 + \cos\theta)$.

19. Find the moment of inertia of the uniform solid in the form of octant of the ellipsoid

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ about OX

20. Prove that the moment of inertia of the area included between the curves $y^2 = 4\,ax$ and $x^2 = 4\,ay$ about the *x*-axis is $\frac{144}{35} M\,a^2$, where *M* is the mass of area included between the curves.

21. A solid body of density *p* is the shape of solid formed by revolution of the cardioid $r = a\,(1 + \cos\theta)$ about the initial line. Show that its moment of inertia about a straight line through the pole perpendicular to the initial line is $\left(\frac{352}{105}\right)\pi l a^5$.

22. Find the product of inertia of a disc in the form of a quadrant of a circle of radius '*a*' about bounding radii.

23. Show that the principal axes at the origin of the triangle enclosed by $x = 0, y = 0, \frac{x}{a}+\frac{y}{b}=1$ are inclined at angles α and $\alpha+\frac{\pi}{2}$ to the *x*-axis, where $a = \frac{1}{2}\tan^{-1}\left(\frac{ab}{a^2-b^2}\right)$

Choose the correct answer:

24. The triple integral $\iiint_T dx\,dy\,dz$ gives

(*i*) Volume of region *T* (*ii*) Surface area of region *T*

(*iii*) Area of region *T* (*iv*) Density of region *T*.

25. The volume of the solid under the surface $az = x^2 + y^2$ and whose base *R* is the circle $x^2 + y^2 = a^2$ is given as

(*i*) $\frac{\pi}{2a}$ (*ii*) $\frac{\pi a^3}{2}$

(*iii*) $\frac{4}{3}\pi a^3$ (*iv*) None of the above. [*U.P., I. Sem. Dec. 2008*]

ANSWERS

1. P	**2.** $\frac{4k}{15}a^5\left(\frac{\pi}{2}-\frac{8}{15}\right)$	**3.** $\frac{k\pi a^4}{16}$	**5.** $\frac{21\pi k a^4}{32}$
6. $\left(\frac{a}{4},\frac{b}{4},\frac{c}{4}\right)$.	**7.** $\left(0,0,\frac{4}{3}\right)$	**8.** $\left(0,0,\frac{a}{3}\right)$	**9.** $\bar{z}=\frac{64\,ma}{45\pi}$
11. $h+\frac{b^2}{4h}$	**13.** $\left(\frac{3a}{8},\frac{3\pi a}{16}\right)$	**15.** $\frac{2}{5}k\pi a^5 h$	
16. $(i)\ \frac{\pi h r^4}{10}\ (ii)\ \frac{\pi h r^2}{60}(2h^2+3r^2)$		**17.** $\frac{35\pi a^4}{16}$	**19.** $\frac{M}{5}(b^2+c^2)$
22. $\rho\frac{a^4}{4}$	**24.** (i)	**25.** (ii)	

CHAPTER 34

Theory of Errors

34.1 NUMBERS

There are two types of numbers

(i) Exact (ii) Approximate

For example; Exact numbers are 1, 3, 5, 7, 10, $\frac{5}{2}$, 6.23.

Approximate numbers are $\frac{4}{3}$ = 1.3333..........

$\sqrt{2}$ = 1.414213..........

π = 3.141592..........

The value of the left hand side can not be expressed by a finite number of digits.

Approximate value of $\frac{4}{3}$ = 1.3333

App. value of $\sqrt{2}$ = 1.4142

and the app. value of π = 3.1416

34.2 SIGNIFICANT FIGURES

The digits used to express a number are called significant digits (figures).

8123, 3.187, 0.8725, contains 4 significant figures. While the numbers 0.0163, 0.00127, 0.000365 and 0.0000345 contain only three significant figures (digits).

Since zeroes before decimal and after decimal only helps to fix the position of decimal point.

Similarly, the numbers 52000 and 8700.00 have two significant figures only.

34.3 ROUNDING OFF

These are number with larger number of digits.

For example; $\frac{22}{7}$ = 3.14285143

In practice it is convenient to limit such number as 3.14 or 3.143.

The dropping of the digits is called rounding off.

Rule: (1) To round of a number to n significant numbers ignore all the digits to the right of nth digit if there is some digit ignore it.

(2) Less than half a unit leave this unit.

(3) Greater than half unit is taken as full unit.

(4) Exactly half unit is taken as one unit in the case of odd numbers *i.e.*, increased the odd number by one. If *n*th number is even, then *n*th number should not be changed.

5.783 to 5.78	7.767 to 7.77
15.976 to 15.9	95767 to 95800
8.4365 to 8.44	87.656 to 87.6

Also the numbers 7.284359, 15.864651, 9.464762 rounded off to four places of decimals at 7.2844, 15.8646, 9.4648 respectively.

34.4 TYPES OF ERRORS

(1) Absolute Errors

The error is defined as a quantity which is added to true value in order to obtain the measured value.

True value + Error = Measured value/observed value.

Correction. The error with sign changed is called correction.

Measured value + Correction = True value.

If x is the true value and X' is approximate value then $|X - X'|$ is called the absolute error.

(2) **Relative Error**

$$\text{Relative error} = \left| \frac{X - X'}{X} \right|$$

(3) **Percentage Error**

$$\text{Percentage error} = \frac{100\,|X - X'|}{X}$$

(4) **Inherent Error**

Errors which are already in data for calculation of a problem before its solution are called inherent error. Such error arrise due to limitation of mathematical tables or the digital computer.

(5) **Rounding off the errors**

Such error arrise by the process of rounding off the numbers. Such errors are un avoidable most of the calculation.

(6) **Truncation Error**

Truncation error are caused by using approximate result on replacing an infinite series.

For example; if $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \infty = X$ (say)

is replaced by $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} = X'$ (say)

then the truncation error $= X - X'$

Notes : (1) If a number is correct to n decimal places then the error is $= \frac{1}{2} 10^{-n}$.

For example **:** If the number is $\sqrt{2}$ = 1.1414 correct to four decimal places, then the error = $\frac{1}{2} \times 10^{-4}$

(2) If the first significant figure of a number is λ and the number is correct to n significant figures, then the relative error is less than $\frac{1}{\lambda \times 10^{n-1}}$.

Verification. 974.16 is correct to five significant figure.

Here, $\lambda = 9,\ n = 5$

$$\text{Absolute error} = \frac{0.01}{2} = 0.005$$

$$\text{Relative error} < \frac{0.005}{974.16} = \frac{5}{974160} = \frac{1}{2 \times 97416}$$

$$< \frac{1}{2 \times 90000} = \frac{1}{2 \times 9 \times 10^4}$$

$$< \frac{1}{9 \times 10^4}$$

$$i.e.,\ \frac{1}{\lambda \times 10^{n-1}}.$$

Example 1. *Round off the numbers 754126 and 16.73117 to four significant figures. Compute absolute error relative error and percentile error.*

Solution. Number rounded off to 4 significant figure equal to 754100

$$\text{Absolute error} = |X - X'| = |754126 - 754100| = |26| = 26$$

$$\text{Relative error} = \left|\frac{X - X'}{X}\right| = \frac{26}{754126} = 3.45 \times 10^{-5}$$

$$\text{Percentile error} = \frac{|X - X'|}{X} \times 100$$

$$= 3.45 \times 10^{-5} \times 100 = 3.45 \times 10^{-3}$$

(*ii*) Number rounded off to four significant figure is 16.73

$$\text{Absolute error} = |X - X'| = |16.73117 - 16.73| = 0.00117$$

$$\text{Relative error} = \left|\frac{X - X'}{X}\right| = \frac{0.00117}{16.73117} = 6.99 \times 10^{-5}$$

$$\text{Percentile error} = \left|\frac{X - X'}{X}\right| \times 100 = 6.99 \times 10^{-5} \times 100 = 6.99 \times 10^{-3}$$ **Ans.**

EXERCISE 34.1

Round off the following numbers correct to three significant figures :

1. 0.0031614

2. 16.132102

3. 0.30617

4. 2945567

5. 45.56735

6. 5.26521

7. Find the relative error if $\frac{1}{3}$ is approximated to 0.334.

8. Find the percentage error if 625.483 is approximated to three significant figures.

9. $\sqrt{29}$ = 5.385 and π = 3.317 correct to four significant figures. Find the relative errors in their sum and difference.

ANSWERS

1. 0.00316	**2.** 16.1	**3.** 0.306
4. 2940000	**5.** 45.6	**6.** 5.26
7. 0.002	**8.** 0.077	**9.** 1.149×10^{-4}, 4.836×10^{-4}

34.5 ERROR DUE TO APPROXIMATION OF THE FUNCTION

Let $z = f(x, y)$ be a function of two variables x and y.

If δx, δy be the errors in x and y, then the error in z is given by $z + \delta z = f(x + \delta x, y + \delta y)$.

Expanding $f(x, y)$ by Taylor's series, we get

$$z + \delta z = f(x, y) + \left(\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y\right) + \text{terms involving higher powers of } \delta x \text{ and } \delta y. \quad ...(1)$$

If δx and δy be so small that their squares and higher powers can be neglected, then (1) can be written as

$$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y \qquad \text{(app.)}$$

In general, if $z = f(x_1, x_2, ... x_n)$ and there are errors in $x_1, x_2 ..... x_n$, then

$$\delta z = \frac{\partial z}{\partial x_1}dx_1 + \frac{\partial z}{\partial x_2}dx_2 + \frac{\partial z}{\partial x_3}dx_3 + ... + \frac{\partial z}{\partial x_n}dx_n.$$

Example 1. *If* $u = \frac{5x^3y^4}{z^5}$ *and errors in x, y, z be 0.001, and compute the relative maximum error when x =1, y = 1, z = 1.*

Solution.
$$u = \frac{5x^3y^4}{z^5} \qquad ...(1)$$

$$\delta x = \delta y = \delta z = 0.001$$

and
$$x = y = z = 1$$

Differentiating (1) partially, with respect to 'x', we get

$$\frac{\delta u}{\delta x} = \frac{15x^2y^4}{z^5}, \quad \frac{\delta u}{\delta y} = \frac{20x^3y^3}{z^5}, \quad \frac{\delta u}{\delta z} = -\frac{25x^3y^4}{z^6}$$

Now, we know that

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z$$

$$= \frac{15x^2y^4}{z^5}\delta x + \frac{20x^3y^3}{z^5}\delta y - \frac{25x^3y^4}{z^6}\delta z$$

The error being maximum

$$(\delta u)_{\max} = \left|\frac{15x^2y^4}{z^5}\delta x\right| + \left|\frac{20x^3y^3}{z^5}\delta y\right| + \left|\frac{25x^3y^4}{z^6}\delta z\right|$$

$$= \left|\frac{15(1)(1)}{(1)}(0.001)\right| + \left|\frac{20(1)(1)}{(1)}(0.001)\right| + \left|\frac{25(1)(1)}{(1)}(0.001)\right|$$

$$= 0.015 + 0.020 + 0.025 = 0.06$$

Relative error $= \dfrac{(\delta u)_{\max}}{u} = \dfrac{0.06}{5} = 0.012$ **Ans.**

Example 2. *Find the maximum error in magnitude in the approximation*

$f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$ *over the rectangle* $R : |x - 3| < 0.01$ *and* $|y - 2| < 0.01$.

Solution. Here, we have

$$f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$$

$$\frac{\partial f}{\partial x} = 2x - y, \quad \frac{\partial f}{\partial y} = -x + y.$$

We know that

$$\text{Maximum } \delta f = \left|\frac{\partial f}{\partial x}\delta x\right| + \left|\frac{\partial f}{\partial y}\delta y\right|$$

$$= |(2x - y)\delta x| + |(-x + y)\delta y|$$

$$= |(2 \times 3 - 2)(0.01)| + |(-3 + 2)\, 0.01|$$

$$= 4(0.01) + |-0.01| = 0.05$$ **Ans.**

34.6 ERROR IN A SERIES APPROXIMATION

By Taylor series of one variable

$$f(x) = f(a + \overline{x - a}) = f(a) + (x - a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a) + R_n(x)$$

Here $R_n(x) = \dfrac{(x-a)^n}{n!}f^n(\theta), \; a < \theta < x$

For a convergent series $R_n(x) \to 0$ as $n \to \infty$.

Approximate value of series = First n terms of the series.

We can find the number of terms for a particular desired accuracy.

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Example 1. *Correct to five places of decimal at $x = 1$ find the number of terms of the approximate series of e^x.*

Solution. We know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n-1}}{(n-1)!} + R_n(x)$$

Here $$R_n(x) = \frac{x^n}{n!}e^{\theta}, 0 < \theta < x$$

Maximum absolute error at $(\theta = x) = \dfrac{x^n}{n!} f^n(x) = \dfrac{x^n}{n!} e^x$

Maximum error at $(x = 1) = \dfrac{1}{n!}$ $(x = 1)$

Maximum error correct to five decimal places $\dfrac{1}{n!} < \dfrac{1}{2} 10^{-5}$.

$$\Rightarrow \qquad n! > 2 \times 10^5 \qquad (8! = 40320)$$

$$8! > 2 \times 10^5 \Rightarrow n = 8$$

Hence there are 8 terms in order that the sum is correct to five places of decimal. **Ans.**

EXERCISE 34.2

1. Find the number of term of the approximated series of e^x correct to six decimal places.
2. Find the number of terms in the approximated series of $\log (1 + x)$ at $x = 1$, (log 2) to six decimal places.
3. The fractional error in the measurement of x is 0.001. What is the corresponding error in expansion of e^x.
4. If $R = \dfrac{4xy^2}{z^3}$ and errors in x, y, z be 0.001, show that the maximum relative error at $x = y = z = 1$ is 0.006.

ANSWERS

1. $n = 10$ **2.** $n = 0$

34.7 ORDER OF APPROXIMATION

$$\text{Function} = f(h)$$

Approximate value of function $= \phi(x)$

$$\text{Error} = E(h^n)$$

$$| f(h) - \phi(h) | \le E\, | h^n |$$

$$\text{Order of error} = O(h^n)$$

$$f(h) = \phi(h) + O(h^n)$$

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Example 1. *Write down with fifth order of approximation of* $\frac{1}{1-h}$.

Solution. We know that

$$\frac{1}{1-h} = (1-h)^{-1} = 1 + h + h^2 + h^3 + h^4 + h^5 + h^6 + h^7 + ...$$

$$= 1 + h + h^2 + h^3 + h^4 + 0(h^5)$$

Example 2. *Write down the seventh order of approximation of sin | h |.*

Solution. We know that

$$\sin |h| = h - \frac{h^3}{3!} + \frac{h^5}{5!} - \frac{h^7}{7!} + \frac{h^9}{9!} + ...$$

sin | h | with seventh order of approximation

$$\sin (h) = h - \frac{h^3}{3!} + \frac{h^5}{5!} + O(h^7)$$ **Ans.**

34.8 MOST PROBABLE VALUE AND RESIDUAL

Let true value of a quantity be X.

Their approximate values are $X_1, X_2, X_3, ..., X_n$.

and the corresponding probable errors are $x_1, x_2, x_3, ... x_n$.

$$x_1 = X_1 - X, x_2 = X_2 - X, x_3 = X_3 - X, ..., x_n = X_n - X$$

In fact we cannot get true value of a quantity due to random errors. For practical purposes we take a probable value $\bar{X}$ of a quantity in place of true value. The probable value $\bar{X}$ is the average of $X_1, X_2, X_3, ..., X_n$. $\left[\bar{X} = \frac{X_1 + X_2 + X_3 + ... + X_n}{n}\right]$

We define the residual by.

$$d_1 = X_1 - \bar{X}, d_2 = X_2 - \bar{X}, d_3 = X_3 - \bar{X} ..... d_n = X_n - \bar{X}.$$

$d_1, d_2, d_3 ..... d_n$ are the residual error and $x_1, x_2, x_3, ... x_n$ are the probable error.

34.9 GAUSSIAN ERROR

Errors and residuals are neither systematic nor constants but equally likely to be positive or negative.

Small errors are more frequent than large ones.

Very large errors don't occur at all.

Under these conditions the errors follow the law of probability given by normal distribution.

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \qquad (\mu = 0)$$

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On putting y = N, $h^2 = \frac{1}{2\sigma^2}$

$$N = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}$$

This given the relative number of measurements of N having error x and h is called precision index.

On plotting a graph between N and x, we get the Gaussian error curve.

34.10 THEORETICAL DISTRIBUTIONS

(1) **Binomial Distribution** $(q + p)^n$

$$P(r) = {}^nC_r p^r q^{n-r}$$

$$\text{Mean} = np$$

$$\text{S.D.} = \sqrt{npq}$$

$$\text{Variance} = npq$$

Mode = Most probable of success = $(n + 1)p$

Recurrence relation, $P(r+1) = \frac{n-r}{r+1}\left(\frac{p}{q}\right).P(r).$

(2) **Poisson' Distribution**

$$P(r) = \frac{e^{-m} m^r}{r!}$$

$$\text{Mean} = m$$

$$\text{S.D.} = \sqrt{m}$$

$$\text{Variance} = m$$

Mode = $[m]$ = Integral part of m $[m - 1 \le r \le m]$

Recurrence relation $P(r+1) = \frac{m}{r+1} P(r).$

(3) **Normal distribution**

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\text{Mean} = \mu$$

$$\text{Standard deviation} = \sigma$$

$$\text{Medium} = 0$$

$$\text{Modal ordinate} = \frac{1}{\sigma\sqrt{2\pi}}.$$

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EXERCISE 34.3

1. Explain the meaning of the terms mean and standard deviation of a term.
2. Calculate the mean deviation and standard deviation of the series

   $a, a + d, a + 2d, ... , a + 2nd$
3. Explain what do you mean by binomial distribution. Find its mean and standard deviation.
4. Define Poisson's distribution. Discuss its importance in physics.
5. Calculate mean and standard deviation of Poisson's distribution.
6. Define probability density function for the normal distribution.
7. Define binomial and normal probability distribution and compare them.
8. Assuming that N is large, show that the error in writing $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$ is approximately $\frac{50(n-1)}{N}$ percent of the value of $\sigma_{\bar{x}}$.
9. State and prove the normal law of errors and find an expression of the measure of precision and the probable error of the arithmetic mean **(*D.U. May 2010*).**
10. Derive the normal law of errors and calculate the probable error of an observation.

35

CHAPTER

Probability and Distributions

35.1 PROBABILITY

Probability is a concept which numerically measure the degree of uncertainty and therefore, of certainity of the occurrence of events to happen or Not to happen.

If an event A can happen in m ways, and fail in n ways, all these ways being equally likely to occur, then the probability of the happening of A is

$$= \frac{\text{Number of favourable cases}}{\text{Total number of mutually exclusive and equally likely cases}} = \frac{m}{m+n}$$

and that of its failing is defined as $\frac{m}{m+n}$

If the probability of the Happening = p

and the probability of Not happening = q

then $$p+q = \frac{m}{m+n} + \frac{n}{m+n} = \frac{m+n}{m+n} = 1 \text{ or } p+q=1.$$

For instance, on tossing a coin, the probability of getting a head $= \frac{1}{2}$.

Some Definitions

1. **Exhaustive Events or Sample Space :** The set of all possible outcomes of a single performance of an experiment is exhaustive events or sample space. Each outcome is called a sample point. In case of tossing a coin once, S = (H, T) is the sample space. Two outcomes - Head and Tail - constitute an Exhaustive event because No other outcome is possible.
2. **Random Experiment :** There are experiments, in which results may be altogether different, even though they are performed under identical conditions. They are known as random experiments. Tossing a coin or throwing a die are random experiments.
3. **Trial and Event :** Performing a random experiment is called a trial and outcome is termed as event. Tossing of a coin is a trial and the turning up of head or tail is an event.
4. **Equally likely events :** Two events are said to be 'equally likely', if one of them cannot be expected in preference to the other. For instance, if we draw a card from well-shuffled pack, we may get any card, then the 52 different cases are equally likely.
5. **Independent events :** Two events may be independent, when the actual happening of one does not influence in any way the probability of the happening of the other.

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Example. The event of getting head on first coin and the event of getting tail on the second coin in a simultaneous throw of two coins are independent.

6. **Mutually Exclusive events :** Two events are known as *mutually exclusive*, when the occurrence of one of them excludes the occurrence of the other. For example, on tossing of a coin, either we get head or tail, but not both.
7. **Compound Event :** When two or more events occur in composition with each other, the simultaneous occurrence is called a compound event. When a die is thrown, getting a 5 or 6 is a compound event.
8. **Favourable Events :** The events which ensure the required happening, are said to be favourable events. For example, in throwing a die, to have the even numbers, 2, 4 and 6 are favourable cases.
9. **Conditional Probability :** The probability of happening an event A, such that event B has already happened, is j called conditional probability of happening of A on the condition that B has already happened. It is usually denoted by $P(A/B)$.
10. **Odds in favour of an event and odds against an event:**

    If number of favourable ways = m, number of not favourable events = n

    (*i*) Odds in favour of the event = $\dfrac{m}{n}$, (ii) Odds against the event = $\dfrac{n}{m}$.
11. **Classical Definition of Probability:** If there are n equally likely, mutually, exclusive and exhaustive events of an experiment and m of these are favourable, then the probability of the happening of the event is defined as $\dfrac{m}{n}$.
12. **Expected value:** if $p_1, p_2, p_3 \ldots p_n$ are the probabilities of events $x_1, x_2, x_3 \ldots x_n$ respectively then the expected value

$$E(X) = p_1x_1 + p_2x_2 + p_3x_3 + \ldots + p_nx_n = \sum_{r=1}^{n} p_r x_r$$

13. **Complement of an event.** The complement of an event E with respect to the sample space S is the set of all elements of S; which are not in E. The complement of E is denoted by E' or $\bar{E}$.

$$E \cap \bar{E} \simeq \phi \qquad \text{or} \qquad E \cap E' = \phi$$

$$P(\bar{E}) = 1 - P(E)$$

Probability of an Event

$$P(A) = \frac{\text{Number of outcomes favourable to A}}{\text{Total number of possible outcomes}}$$

Odds. If an event E occurs in m ways and does not occur in n ways, then

(*i*) Odds in favour of the event = $\dfrac{m}{n}$ (*ii*) Odds against the event = $\dfrac{n}{m}$

(*iii*) $P(E) = \dfrac{m}{m+n}$

Addition law of probability. If A and B are two events associated with an experiment, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

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and $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$.

Multiplication law of probability. If A and B are two events associated with a random experiment, then

$$P(A \cap B) = P(A) \times P(B)$$

Combination. Number of combinations of n things taken r at a time is denoted by ${}^nC_r$.

$${}^nC_r = \frac{n!}{r!(n-r)!} \quad \text{and} \quad {}^nC_r = {}^nC_{n-r}$$

35.2 ODDS OF AN EVENT

Odds are closely related to probability.

For example. One card is drawn from a well shuffled deck of 52 cards, find out the probability of an ace, and also find the probability of not ace.

Here, there are 4 aces in a deck of 52 cards. Therefore

$$P\text{ (ace)} = \frac{4}{52} = \frac{1}{13}$$

$$\text{Also P (not ace)} = \frac{52-4}{52} = \frac{48}{52} = \frac{12}{13}$$

Since, the probability of drawing non-ace is 12 times the probability of drawing an ace, we say the odds in favour of an ace are 1 to 12, or alternatively the odds against an ace are 12 to 1.

Therefore, If an event E occurs in m ways and not occur in n ways, then we say that

(*i*) Odds in favour of the event $= \dfrac{m}{n}$ (ii) Odds against the event $= \dfrac{n}{m}$

(*iii*) $P(E) = \dfrac{m}{m+n}$

Odds in favour of an event = prob. (Success): prob. (Failure)

Odds against an event = prob. (Failure): prob. (Success).

In general

$$\text{Odds in favour of an event } E = \frac{P(E)}{1-P(E)} = \frac{p}{1-p}$$

$$\text{Odds against event } E = \frac{1-P(E)}{P(E)} = \frac{1-p}{p}$$

Example. *A card is drawn from a well shuffled deck of 52 cards. What are the odds in favour of getting a face card? What are the odds against getting a face card?*

Solution. There are 12 face cards (kings, queens, and jacks) in a pack of 52 cards. So, the cards other than face cards are (52 – 12) = 40

∴ There are 12 outcomes favourable to the event “a face card” the 40 outcomes are unfavourable.

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⇒ Odds in favour of getting “a face card”

$$= \frac{Number\ of\ \ favourable\ outcomes}{Number\ of\ \ unfavourable\ outcomes} = \frac{12}{40} = \frac{3}{10} \text{ or 3 to 10}$$

∴ Odds against getting “a face card”

$$= \frac{Number\ of\ \ favourable\ outcomes}{Number\ of\ \ unfavourable\ outcomes} = \frac{40}{12} = \frac{10}{3} \text{ or 10 to 3}$$

35.3 CONDITIONAL PROBABILITY

Let A and B be two events of a sample space S and let $P(B) \neq 0$. Then conditional probability of the event A given B, denoted by $P(A/B)$, is defined by

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \qquad \ldots(1)$$

Theorem. If the events A and B defined on a sample space S of a random experiment are independent then

$$P(A/B) = P(A) \text{ and } P(B/A) = P(B)$$

Proof. A and B are given to be independent events,

$$P(A \text{ and } B) = P(A) \,.\, P(B)$$

$$\Rightarrow \qquad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A).P(B)}{P(B)} = P(A)$$

$$\Rightarrow \qquad P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B).P(A)}{P(A)} = P(B)$$ **Proved**

35.4 BAYES' THEOREM

If $B_1, B_2, B_3, \ldots, B_n$ are mutually exclusive events with $P(B_i) \neq 0$, $(i = 1, 2, \ldots n)$ of a random experiment then for any arbitrary event A of the sample space of the above experiment with $P(A) > 0$, we have

$$P(B_i/A) = \frac{P(B_i)P(A/B_i)}{\sum_{i=1}^{n} P(B_i)P(A/B_i)} \qquad \text{(for } n = 3\text{)}$$

$$P(B_2/A) = \frac{P(B_2)P(A/B_2)}{P(B_1)P(A/B_1) + P(B_2)P(A/B_2) + P(B_3)P(A/B_3)}$$

Proof. Let S be the sample space of the random experiment.

The events $B_1, B_2, \ldots, B_n$ being exhaustive

$$S = B_1 \cup B_2 \cup \ldots \cup B_n \qquad [\because A \subset S]$$

$$A = A \cap S = A \cap (B_1 \cup B_2 \cup \ldots \cup B_n)$$

$$\therefore \qquad = (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_n) \qquad \text{[Distributive Law]}$$

$$\Rightarrow \quad P(A) = P(A \cap B1) + P(A \cap B_2) + \ldots + P(A \cap B_n)$$

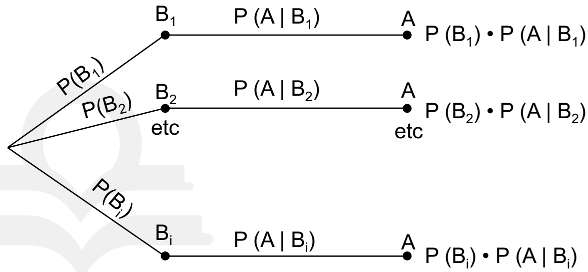
$$= P(B_1)P(A / B_1) + P(B_2)P(A \mid B_2) + \ldots + P(B_n)P(A \mid B_n)$$

$$= \sum_{i=1}^{n} P(B_i)P(A / B_i) \qquad \ldots(1)$$

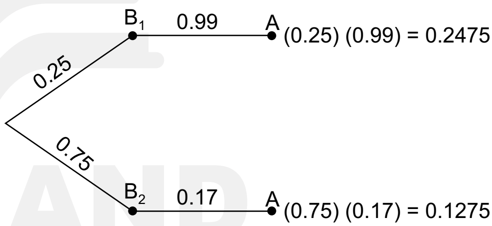
Now $\quad P(A \cap B_i) = P(A\ )P(B_i \mid A)$

$$\Rightarrow \quad P(B_i / A) = \frac{P(A \cap B_i)}{P(A)} = \frac{P(B_i)P(A / B_i)}{\sum_{i=1}^{n} P(B_i)P(A / B_i)} \qquad \text{[using (1)]}$$

Note. $P(B)$ is the probability of occurrence of B. If we are told that the event A has already occurred. On knowing about the event A, $P(B)$ is changed to $P(B \mid A)$. With the help of Bayes' theorem we can calculate $P(B \mid A)$.



Example 1. *In a certain state, 25 percent of all cars emit excessive amounts of pollutants. If the probaility is 0.99 that a car emitting excessive amounts will fail the states vehicular emission test, and the probability is 0.17 that a car not emitting excessive amounts of pollutants will nevertheless fail the test. Whatsis the probability that a car that fails the test actually emits excessive amounts of pollutants?*



Solution. In the diagram we find that the probabilities associated with the two branches of the diagram are (0. 25) (0.99) = 0.2475 and (1 – 0.25) (0.17) = 0.1275. Thus, the probability that a car that fails the test actually emits excessive amounts of pollutants is

$$\frac{0.2475}{0.2475 + 0.1275} = 0.66$$

This result could also have been obtained without the diagram by substituting directly into the formula of Bayes' theorem.

Solution by the formula of Bayes' theorem

Let A is the event that the car will fail the emission test and B_1 is the event that the car emit excessive amount of pollutants and B_2 is the event hat the cars do not emit excessive pollutants.

We have $\quad P(B_1) = 0.25 \quad \therefore \quad P(B_2) = 0.75$

$\qquad P(A \mid B_1) = 0.99 \quad , P(A \mid B_2) = 0.17$

By Bayes' theorem

$$P(B/A) = \frac{P(B_1).P(A/B_1)}{P(B_1).P(A/B_1) + P(B_2).P(A/B_2)} = \frac{(0.25)(0.99)}{(0.25)(0.99) + (0.75)(0.17)}$$

$$= \frac{0.2475}{0.2475 + 0.1275} = 0.66$$

Example 2. *The members of a consulting firm rent cars from three rental agencies: 60 percent from agency I, 30 percent from agency 2, and 10 percent from agency 3. If 9 percent of the cars from agency 1 need a tune-up, 20 percent of the cars from agency 2 need a tune-up, and 6 percent of the cars from agency 3 need a tune-up, what is the probability that a rental car delivered to the firm will need a tune-up?*

If a rental car delivered to the consulting firm needs a tune-up, what is the probability that it came from rental agency 2?

Solution. If A is the event that the car needs a tune-up, and B_1, B_2 and B_3 are the events that the car comes from rental agencies 1, 2, or 3, we have P (B_1) = 0.60, P (B_2) = 0.30, P (B_3) = 0.10, P (A | B_1) = 0.09, P (A | B_2) = 0.20, and P (A | B_3) = 0.06. Substituting these values into the formula.

$$P(A) = \sum_{i=1}^{k} P(B_i).P(A/B_i)$$

$$\Rightarrow \qquad P(A) = (0.60)(0.09) + (0.30)(0.20) + (0.10)(0.06)$$

$$= 0.12$$

Thus, 12% of all the rental cars dilvered to this firm will need a tune - up.

By Bayes' theorem

$$P(B_2/A) = \frac{P(B_2)\cdot P(A/B_2)}{P(B_1)\cdot P(A/B_1) + P(B_2)\cdot P(A/B_2) + P(B_3)\cdot P(A/B_3)}$$

$$= \frac{(0.30)(0.20)}{(0.60)(0.09) + (0.30)(0.20) + (0.10)(0.06)}$$

$$= 0.5$$ **Ans.**

Example 3. *Three urns contains 6 red, 4 black; 4 red, 6 black; 5 red, 5 black balls respectively. One of the urns is selected at random and a ball is drawn from it. If the ball drawn is red find the probability that it is drawn from the first urn.*

[D.U. Dec, 2017]

Solution. Let U_1: the ball is drawn from *urn I*.

U_2: the ball is drawn from *urn II*.

U_3: the ball is drawn from *urn III*.

R: the ball is red.

We have to find $P(U_1/R)$.

By Baye's Theorem,

$$P(U_1/R) = \frac{P(U_1)P(R/U_1)}{P(U_1)P(R/U_1) + P(U_2)P(R/U_2) + P(U_3)P(R/U_3)} \qquad \ldots(1)$$

Since the three urns are equally likely to be selected $P(U_1) = P(U_2) = P(U_3) = \frac{1}{3}$

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Also $P\ (R/U_1) = P$ (a red ball is drawn from urn I) $= \frac{6}{10}$

$P\ (R/U_2) = P$ (a red ball is drawn from urn II) $= \frac{4}{10}$

$P\ (R/U_3) = P$ (a red ball is drawn from urn III) $= \frac{5}{10}$

$\therefore$ From (1), we have = $P(U_1 / R) = \dfrac{\frac{1}{3}\times\frac{6}{10}}{\frac{1}{3}\times\frac{6}{10}+\frac{1}{3}\times\frac{4}{10}+\frac{1}{3}\times\frac{5}{10}} = \frac{2}{5}$ **Ans.**

Example 4. *In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total. If their output 5, 4 and 2 per cent are defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it was manufactured by machine B?* ***[D.U. Dec, 2017]***

Solution. A: bolt is manufactured by machine A.

B: bolt is manufactured by machine B.

C: bolt is manufactured by machine C

$P\ (A) = 0.25,\ P\ (B) = 0.35,\ P\ (C) = 0.40$

The probability of drawing a defective bolt manufactured by machin e A is $P\ (D/A) = 0.05$

Similarly, P $(D/B) = 0.04$ and $P\ (D/C) = 0.02$

By Baye's theorem

$$P(B / D) = \frac{P(B)P(D / B)}{P(A)P(D / A) + P(B)P(D / B) + P(C)P(D / C)}$$

$$= \frac{0.35 \times 0.04}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02} = 0.41$$

EXERCISE 35.1

1. If 20% of the bolts produced by a machine are defective, determine the probability that out of 4 bolts chosen at random

(a) 1 (b) 0 (c) At most 2 bolts will be defective.

2. Six dice are thrown 729 times. How many times do you expect at least three dice to show a five or a six?

3. If the chance that any one of the 10 telephone lines is busy at any instant is 0.2, what is the chance that 5 of the lines are busy ? What is the probability that all the lines are busy?

4. An insurance salesman sells policies to 5 men, alt of identical age in good health. According to the actuarial tables the probability that a man of this particular age will be alive 30 years hence is $\frac{2}{3}$. Find the probability that in 30 years.

(a) All 5 men (b) At least 3 men (c) Only 2 men (d) At least 1 man will be alive.

5. Assuming a Binomial distribution, find the probability of obtaining at least two "six" in rolling a fair die 4 times.

6. If successive trials are independent and the probability of success on any trial is pt show that the probability that the first success occurs on the nth trial is

$$p(1-p)^{n-1}, \qquad n = 1, 2, 3...$$

7. Consider an urn in which 4 balls have been placed by the following scheme : A fair coin is tossed; if the coin falls head, a white ball is placed in the urn, and if the coin falls tail, a red ball is placed in urn. (*i*) What is the probability that the urn will contain exactly 3 white balls ? (*ii*) What is the probability that the urn will contain exactly 3 red balls, given that the first ball placed was red?

8. A box contains 10 screws. 3 of which are defective. Two screws are drawn at random without replacement. Find the probability that none of the two screws is defective.

9. Out of 800 families with four children each, how many families would be expected to have :

(*i*) 2 boys and 2 girls; (*ii*) at least one boy; (*iii*) no girl; (*iv*) at most two girls?

Assume equal probabilities for boys and girls.

10. A fair dice is rolled. Consider the events A = {1, 3, 5}, B = {2, 3} And C = {2, 3, 4, 5}. Find

(*i*) P (A / B) and P (B / A) (*ii*) P (A / B) and P (C / A).

(*iii*) P (A ∪ B / C) and P (A ∩ B / C) **[*D.U. Nov, 2015*]**

11. If A and B are independent events associated with a random experiment, then prove that

(*i*) $\bar{A}$ and $\bar{B}$ are independent events (*ii*) A and $\bar{B}$ are independent events

(*iii*) $\bar{A}$ and $\bar{B}$ are also independent events. **[*D.U. Nov, 2015*]**

ANSWERS

1. (*a*) 0.4096 (*b*) 0.4096 (*c*) 0.9728.

2. 233

3. ${}^{10}C_5\,(0.2)^5\,(0.8)^5,\ (0.2)^{10}$

4. (*a*) $\frac{32}{243}$ (*b*) $\frac{192}{243}$ (*c*) $\frac{40}{243}$ (*d*) $\frac{242}{243}$

5. $\frac{171}{1296}$

7. (*i*) $\frac{1}{8}$ (*ii*) $\frac{3}{8}$

8. $\frac{7}{15}$

9. (*i*) 300 (*ii*) 750 (*iii*) 50 (*iv*) 550.

35.5 RANDOM VARIABLES

A *random variable* is a variable whose possible values are numerical outcomes of a random phenomenon.

Therefore a Random Variable can be defined as a real number '*X*' which is associated with the outcomes of a random experiment. Let us consider the case of single throw of a die, if *X* denotes the number obtained, then *X* is a random variable which can take any value 1, 2, 3, 4, 5 or 6 with equal probability 1/6.

Further, if we consider the three tosses of a coin then the total number of cases will be $2^3 = 8$. And the sample space is given below:

S = {HHH, HHT, HTH, THH. THT, HTT, TTH, TTT}

Let us consider the case of number of Tails, Then *X* is a random variable which may take any value from 0, 1, 2, and 3.

Outcome	HHH	HHT	HTH	THH	THT	HTT	TTH	TTT
Values of X	3	2	2	2	1	1	1	0

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35.6 DISCRETE AND CONTINUOUS RANDOM VARIABLES

(a) Discrete Random Variable simply defines a set consisting of finite or countable set of values. *Discrete Random Variable* may take only a countable number of distinct values such as 0,1,2,3,4….. Discrete random variables are usually (but not necessarily) counts. If a random variable can take only a finite number of distinct values, then it must be discrete. Examples of discrete random variables include the number of children playing in park, the number of voters of a particular city, the number of students in a school etc. are all discrete random variables.

(b) Continuous Random Variable : Defines a set which consists of infinite and un-countable f set of values. For example, the age, height or weight of students in a class are all continuous I random variables. Generally, we can say that random variables represent counted data while continuous random variables represent measured data. For example, Random Variables like Length, Thickness, Weights and Temperature are called Continuous Variables.

35.7 PROBABILITY DISTRIBUTION FUNCTION

Definition. Let X be a *random variable. The function F defined for all real x by*

$$F(x) = P(X \leq x) = P\{\omega : X(\omega) \leq x\}, \; -\infty < x < \infty, \quad \text{...(1)}$$

is called the distribution function (d.f.) of the r.v. (X).

Remark. A distribution function is also called the cumulative distribution function. Sometimes, the notation $F_X(x)$ is used to emphasise the fact that the distribution function is associated with the particular random variable X. Clearly, the domain of the distribution function is $(-\infty, \infty)$ and its range is $[0, 1]$.

35.8 PROPERTIES OF DISTRIBUTION FUNCTION

We now proceed to derive a number of properties common to all distribution functions.

1. *If F is the d.f. of the random variable X and if $a < b$, then $P(a < X \leq b) = F(b) - F(a)$.*

Proof. The events '$a < X \leq b$' and '$X \leq a$' are disjoint and their union is the event '$X \leq b$'., Hence by addition theorem of probability :

$P(a < X \leq b) + P(X \leq a) = P(X \leq b)$

$$\Rightarrow \quad P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a) \quad \text{...(2)}$$

Cor. l. $P(a \leq X \leq b) = P\{(X = a) \cup (a < X \leq b)\} = P(X = a) + P(a < X \leq b)\}$

$$= P(X = a) + [F(b) - F(a)] \quad \text{... (3)}$$

Similarly, we get

$$P(a < X < b) = P(a < X \leq b) - P(X = b) = F(b) - F(a) - P(X = b) \quad \text{...(4)}$$

$$P(a \leq X < b) = P(a < X < b) + P(X = a)$$

$$= F(b) - F(a) - P(X = b) + P(X = a) \quad \text{... (5)}$$

Remark. When $P(X = a) = 0$ and $P(X = b) = 0$, all four events $a \leq X \leq b$, $a < X < b$, $a \leq X < b$ and $a < X \leq b$ have the same probability $F(b) - F(a)$.

2. *If F is d.f. of one-dimensional random variable X, then (i) $0 \leq F(x) \leq 1$,*
(ii) $F(x) \leq F(y)$ if $x \leq y$.

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In other words, all distribution functions are monotonically non-decreasing and lie between 0 and 1.

3. *If F is d.f. of one-dimensional r.v. X, then*

$$F(-\infty) = \lim_{x \to -\infty} F(x) = 0 \text{ and } F(\infty) = \lim_{x \to \infty} F(x) = 1$$

Proof. Let us express the whole sample space S as a countable union of disjoint events as follows:

$$S = \left\{ \bigcup_{n=1}^{\infty} (-n < X \le -n+1) \right\} \cup \left\{ \bigcup_{n=0}^{\infty} (n < X \le n+1) \right\}$$

$$\Rightarrow \qquad P(S) = \sum_{n=1}^{\infty} P(-n < X \le -n+1) + \sum_{n=0}^{\infty} P(n < X \le n+1)$$

$$\Rightarrow \qquad 1 = \lim_{a \to \infty} \sum_{n=1}^{a} \{F(-n+1) - F(-n)\} + \lim_{b \to \infty} \sum_{n=0}^{b} \{F(n+1) - F(n)\}$$

$$= \lim_{a \to \infty} \{F(0) - F(-a)\} + \lim_{b \to \infty} \{F(b+1) - F(0)\}$$

$$= \{F(0) - F(-\infty)\} + \{F(\infty) - F(0)\}$$

$$\therefore \qquad 1 = F(\infty) - F(-\infty) \qquad \ldots(A)$$

Since $-\infty < \infty$, $F(-\infty) \le F(\infty)$. Also $F(-\infty) \ge 0$ and $F(\infty) \le 1$

$$\therefore \qquad 0 \le F(-\infty) \le F(\infty) \le 1 \qquad \ldots(B)$$

From (A) and (B), we get F $(-\infty) = 0$ and F $(\infty) = 1$.

Remarks 1. Discontinuities of $F(x)$ are at most countable.

2. $$F(a) - F(a-0) = \lim_{h \to 0} P(a - h \le X \le a), h < 0$$

and $$F(a+0) - F(a) = \lim_{h \to 0} P(a \le X \le a + h) = 0, h > 0$$

35.9 DISCRETE RANDOM VARIABLE

Simply defines a set consisting of finite or countable set of values. *Discrete Random Variable* may take only a countable number of distinct values such as 0, 1, 2, 3, 4….. Discrete random variables are usually (but not necessarily) counts. If a random variable can take only a finite number of distinct values then it must be discrete. Example of

35.10 PROBABILITY MASS FUNCTION (p.m.f)

Let us consider a random variable X which assumes the values $x_1, x_2, \ldots x_n$. With each value of the variable X, we associate a number

$$P_i = P(X = X_i);\ i = 1, 2, \ldots n.$$

which is known as the probability of x-. and satisfies the following conditions

(*i*) $P_i = P(X = X_i) \ge 0$ $(i = 1, 2, \ldots n)$ (*ii*) $\Sigma P_i = P_1 + P_2 + \ldots . P_n = 1$

The set of all the possible ordered pairs $\{x, p(x)\}$, is called probability distribution of the random variable X.

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The *probability distribution* of a discrete random variable is a list of probabilities associated with each of its possible values. It is also sometimes called the "Probability Function or Probability Mass Function (p.m.f)". pmf is usually to define a Discrete Probability Distribution for either Scalar or Multivariate Random Variables whose domain is discrete.

35.11 DISCRETE PROBABILITY DISTRIBUTION

A Discrete Probability Distribution describes the Probability of occurrence of each value of a discrete random variable. Therefore discrete random variable is a random variable that has countable values, i.e. a list of non-negative integers. When a *random variable* is a discrete variable, its *probability distribution* is called a discrete probability distribution.

With a Discrete Probability Distribution, each possible value of the discrete random variable can be associated with a non-zero probability. Hence a discrete probability distribution is usually presented in tabular form.

Suppose a discrete variate X is the outcome of some random experiment. The probability that X takes the values x_i is p_i then

$$P(X = x_i) = p_i \text{ or } p(x_i) \text{ for } i = 1, 2, \ldots,$$

where (*i*) $p(x_i) > 0$ for all values of i, (*ii*) $\Sigma p(x_i) = 1$.

The set of values x_i with their probabilities p_i constitute *discrete probability distribution* of the discrete variate X.

Example. *A random variable X has the following probability function:*

x :	0	1	2	3	4	5	6	7
$p(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(*i*) Find the value of the k (*ii*) Evaluate $P(X < 6)$, $P(X \geq 6)$ (*iii*) $P(0 < X < 5)$

Solution: (*i*) If X is a random variable,

then $$\sum_{i=0}^{7} p(x_i) = 1, \textit{ i.e. } 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

i.e. $$9k + 10k^2 = 1,\ k = \frac{1}{10}$$

(*ii*) $P(X < 6) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$

$$= 0 + k + 2k + 2k + 3k + k^2 = 8k + k^2 = \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$$

$$P(X \geq 6) = P(X = 6) + P(X = 7) = 2k^2 + 7k^2 + k = \frac{9}{100} + \frac{1}{10} = \frac{19}{100}$$

(*iii*) $P(0 < X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = k + 2k + 2k + 3k$

$$= 8k = \frac{8}{10} = \frac{4}{5}$$ **Ans.**

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EXERCISE 35.2

1. If X be a random variable giving the number of aces in a random draw of 4 cards from an ordinary deck of 52 cards. Plote a table of the probability distribution of X.

2. On the day of rains, a raincoat seller can earn 500 per day. With no rains, he can lose 100 per day. What is his Expectation if the probability of rains is 0.4?

3. A die is throws at random. Calculate the expectation of the number on it

4. A and B throw one dice for a price of ₹ 11 which is to be won by the player who first throws 6. If A has the first throw, what are their respective expectations?

5. A coin is tossed three times; If X is a random variable giving the number of tails that appear, make a table showing the probability distribution of X.

6. A random variable X has the following probability distributions:

X:	0	1	2	3	4	5	6	7
P(X):	0	k	$2k$	$2k$	$3k$	$k2$	$2k^2$	$7k^2 + k$

Find each of the following:

(*i*) k (*ii*) $P(X < 6)$ (*iii*) $P(X \geq 6)$ (*iv*) $P(0 < \times < 5)$

7. The probability that there is at least one error in an accounts statement prepared by A is 0.2 and for B and C they are 0.25 and 0.4, respectively. A, B and C prepared 10, 16 and 20 statements, respectively. Find the expected number of correct statements in all.

8. A box has 5 Blue and 3 Red balls. If 2 balls are to be drawn at random without replacement and X denotes the number of Blue balls, find the probability distribution for X.

9. The probability of a man hitting target is 1/2. How many times must he fire so that the probability of hitting the target at least once is more than 90%.

10. Suppose X has a binomial distribution with parameters n and P. For what P is VAR maximized if n is fixed. Also find the maximum value of variance.

ANSWERS TO SELECTED QUESTIONS

1.

x	0	1	2	3	4
$f(x)$	$\frac{194580}{270725}$	$\frac{69184}{270725}$	$\frac{6768}{270725}$	$\frac{192}{270725}$	$\frac{1}{270725}$

2. ₹140 **3.** $\frac{7}{2}$ **4.** ₹6, ₹ 5

5.

x	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

6. (*i*) $k = \frac{1}{10}$ (*ii*) $\frac{81}{100}$ (*iii*) $\frac{19}{100}$ (*iv*) $\frac{4}{5}$

7. 32

8.

x	0	1	2
$f(x)$	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{5}{28}$

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35.12 BINOMIAL DISTRIBUTION P (r) = $^nc_r\, p^r .q^{n-r}$

To find the probability of the happening of an event once, twice, thrice, ….. ***r*** times exactly in ***n*** trials

Let the probability of the happening of an event ***A*** in one trial be ***p*** and its probability of not happening be $1 - p = q$. We assume that there are ***n*** trials and the happening of the event ***A*** is ***r*** times and its not happening is $n - r$ times. This may be shown as follows

$$\underset{r \text{ times}}{AA.....A} \qquad \underset{n-r \text{ times}}{\bar{A}\bar{A}\;.......\;\bar{A}} \qquad ...(1)$$

A indicates its happening, $\bar{A}$ its failure and $P(A) = p$ and $P(\bar{A}) = q$

We see that (1) has the probability

$$\underset{r \text{ times}}{pp......p} \qquad \underset{n-r \text{ times}}{q.q\;......\;q} = p^r \,.\, q^{n-r} \qquad ...(2)$$

Clearly (1) is merely one order of arranging $r\ A's$:

The probability of (1) $= p^r q^{n-r} \times$ Number of different arrangements of $r\ A$'s and $(n - r)\ \bar{A}'s$.

The number of different arrangements of $r\ A's$ and $(n - r)\ \bar{A}'s = {}^nC_r$.

$\therefore$ Probability of the happening of an event r times $= {}^nC_r\, p^r\, q^{n-r}$.

$= (r + 1)$th term of $(q + p)^n$

$(r = 0, 1, 2, ….. , n)$.

If $r = 0$, probability of happening of an event 0 times $= {}^nC_o\, q^n\, p^0 = q^n$

If $r = 1$, probability of happening of an event 1 time $= {}^nC_1 q^{n-1}\, p$

If $r = 2$, probability of happening of an event 2 times $= {}^nC_2\, q^{n-2}\, p^2$

If $r = 3$, probability of happening of an event 3 times $= {}^nC_3\, q^{n-3}\, p^3$ and so on.

These terms are clearly the successive terms in the expansion of $(q + p)^n$.

Hence, it is called Binomial Distribution.

Example 1. *Find the probability of getting 4 heads in 6 tosses of a fair coin.*

Solution. $P = \frac{1}{2}$, $q = \frac{1}{2}$, $n = 6$, $r = 4$.

We know that $P(r) = {}^nC_r q^{n-r}\, p^r \Rightarrow$ P (4) $= {}^6C_4 q^{6-4} p^4$

$$= \frac{6\times 5}{1\times 2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4 = 15\times\left(\frac{1}{2}\right)^6 = \frac{15}{64} \qquad \textbf{Ans.}$$

Mean of Binomial Distribution

$$\boxed{\text{Mean} = \frac{\sum fr}{\sum f} = np}$$

Variance and Standard Deviation of Binomial Distribution

$$\boxed{\text{Variance} = \sigma^2 = n\,p\,q} \qquad \boxed{S.D. = \sigma = \sqrt{n\,p\,q}}$$

Hence for the binomial distribution, Mean $= np$, and $\mu_2 = \sigma^2 = n\,p\,q$

Example 2. *An urn contains nine balls, two of which are red three blue and four black. Three balls are drawn from the urn at random. What is the probability that*

(*i*) *the three balls are of different colours?* (*ii*) *the three balls are of the same colour?*

Solution.

Urn contains 2 Red balls, 3 Blue balls and 4 Black balls.

(*i*) Three balls will be of different colours if one ball is red, one blue and one black ball are drawn.

$$\text{Required probability} = \frac{{}^2C_1 \times {}^3C_1 \times {}^4C_1}{{}^9C_3} = \frac{2\times 3\times 4}{84} = \frac{2}{7} \qquad \textbf{Ans.}$$

(*ii*) Three balls will be of the same colour if either 3 blue balls or 3 black balls are drawn.

P (3 Blue balls or 3 Black balls) = P (3 Blue balls) + P (3 Black balls)

$$= \frac{{}^3C_3}{{}^9C_3} + \frac{{}^4C_3}{{}^9C_3} = \frac{1+4}{84} = \frac{5}{84} \qquad \textbf{Ans.}$$

Example 3. *An urn A contains 2 white and 4 black balls. Another urn B contains 5 white and 7 black balls. A ball is transferred from the urn A to the urn B, then a ball is drawn from urn B. Find the probability that it is white*:

Solution. Urn A contains 2 white and 4 black balls.

Urn B contains 5 white and 7 black balls.

Now there are two cases of transferring a ball from A to B.

Case I. When a white ball is transferred from A to B

$$P\text{ (Transfer of a white ball)} = \frac{2}{2+4} = \frac{1}{3}$$

After transfer of a white ball, urn B contains 6 white balls and 7 black balls.

P (Drawing a white ball from urn B after transfer)

$$= P\text{ (Transfer of a white ball)} \times P\text{ (Drawing of a white ball)}$$

$$= \left(\frac{1}{3}\right)\left(\frac{6}{6+7}\right) = \frac{1}{3}\times\frac{6}{13} = \frac{2}{13}$$

Case II. When a black ball is transferred from A to B.

$$P\text{ (Transfer of a black ball)} = \frac{4}{2+4} = \frac{2}{3}$$

After transfer of a black ball, urn B contains 5 white and 8 black balls.

P (Drawing a white ball from urn B after transfer)

$$= P\text{ (Transfer of a black ball)} \times P\text{ (Drawing of a white ball)}$$

$$\text{Required probability} = \frac{2}{13} + \frac{10}{39} = \frac{16}{39} \qquad \textbf{Ans.}$$

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Example 4. *A can hit a target 3 times in 5 shots, B 2 times in 5 shots and C three times in 4 shots. All of them fire one shot each simultaneously at the target. What is the probability that*

(*i*) *2 shots hit* (*ii*) *At least two shots hit?* **(*D.U. Dec 2017*)**

Solution. Probability of A hitting the target = $\frac{3}{5}$

Probability of B hitting the target = $\frac{2}{5}$

Probability of C hitting the target = $\frac{3}{4}$

Probability that 2 shots hit the target

$$= P(A)\,P(B)\,q(C) + P(A)\,P(C)\,q(B) + P(B)\,P(C)\,q(A)$$

$$= \frac{3}{5}\times\frac{2}{5}\times\left(1-\frac{3}{4}\right)+\frac{3}{5}\times\frac{3}{4}\times\left(1-\frac{2}{5}\right)+\frac{2}{5}\times\frac{3}{4}\times\left(1-\frac{3}{5}\right)$$

$$= \frac{6}{25}\times\frac{1}{4}+\frac{9}{20}\times\frac{3}{5}+\frac{6}{20}\times\frac{2}{5}=\frac{6+27+12}{100}=\frac{45}{100}=\frac{9}{20}$$ **Ans.**

(*ii*) Probability of at least two shots hitting the target

= Probability of 2 shots + probability of 3 shots hitting the target

$$= \frac{9}{20}+P(A)\,P(B)\,P(C)=\frac{9}{20}+\frac{3}{5}\times\frac{2}{5}\times\frac{3}{4}=\frac{63}{100}$$ **Ans.**

Example 5. *A and B throw alternatively a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. Find their respective chances of winning, if A begins.*

Solution. Number of ways of throwing 6

i.e. (1 + 5), (2 + 4), (3 + 3), (4 + 2), (5 + 1) = 5.

Probability of throwing 6 = $\frac{5}{36} = p_1$, $\quad q_1 = \frac{31}{36}$

Number of ways of throwing 7

i.e. (1 + 6), (2 + 5), (3 + 4), (4 + 3), (5 + 2), (6 + 1) = 6

Probability of throwing $6 = \frac{6}{36} = \frac{1}{6} = P_2, \quad q_2 = \frac{5}{6}$

$$P(A) = p_1 + q_1 q_2 p_1 + q_1^2 q_2^2 p_1 + ...$$

$$P(B) = q_1 p_2 + q_1^2\ q_2\ p_2 + q_1^3 q_2^2 p_2 + ...$$

Probability of A's winning = $p_1 + q_1 q_2 p_1 + q_1^2 q_2^2 p_1 + ...$

$$= \frac{p_1}{1-q_1 q_2} = \frac{\frac{5}{36}}{1-\frac{31}{36}\times\frac{5}{6}} = \frac{5}{6}\times\frac{36\times 6}{61} = \frac{30}{61}$$

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Probability of B's winning = $q_1 p_2 + q_1^2\, q_2\, p_2 + q_1^3 q_2^2 p_2 + ...$

$$= \frac{q_1 p_2}{1 - q_1 q_2} = \frac{\frac{31}{36} \times \frac{1}{6}}{1 - \left(\frac{31}{36}\right)\left(\frac{5}{6}\right)} = \frac{31}{36 \times 6} \times \frac{36 \times 6}{61} = \frac{31}{61}$$ **Ans.**

EXERCISE 35.3

1. If 20% of the bolts produced by a machine are defective, determine the probability that out of 4 bolts chosen at random bolts will be defective.

(*a*) 1 (*b*) 0 (*c*) At most 2

2. Six dice are thrown 729 times. How many times do you expect at least three dice to show a five or a six ?

3. If the chance that any one of the 10 telephone lines is busy at any instant is 0.2, what is the chance that 5 of the lines are busy ? What is the probability that all the lines are busy?

4. An insurance salesman sells policies to 5 men, all of identical age in good health. According to the actuarial tables the probability that a man of this particular age will be alive 30 years hence is $\frac{2}{3}$. Find the probability that in 30 years.

(*a*) All 5 men (*b*) At least 3 men (*c*) Only 2 men (*d*) At least 1 man will be alive.

5. Assuming a Binomial distribution, find the probability of obtaining at least two "six" in rolling a fair die 4 times.

6. If successive trials are independent and the probability of success on any trial is p, show that the probability that the first success occurs on the nth trial is

$$p(1 - p)^{n-1}, \qquad n = 1, 2, 3 ...$$

7. Consider an urn in which 4 balls have been placed by the following scheme : A fair coin is tossed; if the coin falls head, a white ball is placed in the urn, and if the coin falls tail, a red ball is placed in urn. (i) What is the probability that the urn will contain exactly 3 white balls ? (ii) What is the probability that the urn will contain exactly 3 red balls, given that the first ball placed was red?

8. A box contains 10 screws, 3 of which are defective. Two screws are drawn at random without replacement. Find the probability that none of the two screws is defective.

9. Out of 800 families with four children each, how many families would be expected to have :

(*i*) 2 boys and 2 girls; (*ii*) at least one boy; (*iii*) no girl; (*iv*) at most two girls?

Assume equal probabilities for boys and girls.

10. In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is 5/6. What is the probability that he will knock down less than 2 hurdles ?

11. An electronic component consists of three parts. Each part has probability 0.99 of performing satisfactorily. The component fails if 2 or more parts do not perform satisfactorily. Assuming that the parts perform independently, determine the probability that the component does not perform satisfactorily.

12. Find the binomial distribution whose mean is 5 and variance is 10/3.

13. The probability that, on joining Engineering College, a student will successfully complete the course of studies is $\frac{3}{5}$. Determine the probability that out of 5 students joining the college (i) none and (ii) at least two will successfully complete the course.

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14. A carton contains 20 fuses, 5 of which are defective. Three fuses are chosen at random and inspected. What is the probability that at most one defective fuse is found?

15. A bag contains three coins, one of which is coined with two heads, while the other two coins are normal and not biased. A coin is thrown at random from the bag and tossed three times in succession. If heads turn up each time, what is the probability that this is the two-headed coin?

16. In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1,000 such samples, how many would be expected to contain at least 3 defective parts?

17. The incidence of occupational disease in an industry is such that the workers have 20% chance of suffering from it. What is the probability that out of 6 workers 4 or more will catch the disease?

18. If the probability of hitting a target is 10% and 10 shots are fired independently, what is the probability that the target will be hit at least once ?

19. Among 10,000 random digits, find the probability p that the digit 3 appears at most 950 times.

20. A fair coin is tossed 400 times. Using normal approximation to the binomial, find the probability that a head will occur (*a*) more than 180 times and (*b*) less than 195 times.

21. Four coins were tossed 200 times. The number of tosses showing 0, 1, 2, 3 and 4 heads were found to be as under. Fit a binomial distribution to these observed results. Find the expected frequencies.

No. of heads:	0	1	2	3	4
No. of tosses:	15	35	90	40	20

22. A firm plans to bid ₹ 300 per tonne for a contract to supply 1000 tonnes of a metal. It has two competitors A and B and it assumes that the probability that A will bid less than 300/- per tonne is 0.3 and that B will bid less than ₹ 300 per tonne is 0.7. If the lowest bidder gets all the business and the firms bid independently, what is the expected value of business in rupees to the firm. **(*A.M.I.E.T.E. Dec. 2006*)**

23. Fill in the blanks :

(a) A coin is biased so that a head is twice as likely to occur as a tail. If the coin is tossed 3 times, the prob. of getting exactly 2 tails, is……..

(b) The probability of getting number 5 exactly two times in five throws of an unbiased die is……..

(c) A die is thrown 6 times. The probability to get greater than 4 appears at least once is ……..

(d) For what, one should be?

(*i*) Obtaining 6 at least once in 4 throws of a die.

or (*ii*) obtaining a double-six at least once in 24 throws with two dice.

(e) The probability of producing a defective bolt is 0.1. The probability that out of 5 bolts one will be defective is……..

(f) If the probability of hitting a target is 5% and 5 shots are fired independently, the probability that the target will be hit at least once is……..

(g) If *n* and *p* are the parameters of a binomial distribution the standard deviation is ……..

(h) The mean, standard deviation and skewness of Binomial distribution are ..... and .....

(i) If three persons selected at random are stopped on a street, then the probability that all of them were born on Sunday is ……..

ANSWERS

1. (*a*) 0.4096, (*b*) 0.4096, (*c*) 0.9728.

2. 233

3. $^{10}C_5\,(0.2)^5\,(0.8)^5, (0.2)^{10}$

4. (*a*) $\frac{32}{243}$ (*b*) $\frac{192}{243}$ (*c*) $\frac{40}{243}$ (*d*) $\frac{242}{243}$

5. $\frac{171}{1296}$

7. (*i*) $\frac{1}{8}$ (*ii*) $\frac{3}{8}$

8. $\frac{7}{15}$

9. (i) 300, (ii) 750, (iii) 50, (iv) 550.

10. $\frac{8}{3}\left(\frac{5}{6}\right)^9$

11. 0.000298

12. $^{15}C_r\left(\frac{1}{3}\right)^r\left(\frac{2}{3}\right)^{15-r}$

13. (*i*) $\frac{32}{3125}$ (*ii*) $\frac{2853}{3125}$

14. $\frac{27}{32}$

15. $\frac{4}{5}$

16. 324

17. $\frac{53}{3125}$

18. $1 - (0.9)^{10} = 0.65$ nearly

19. $^{10,000}C_r\left(\frac{1}{10}\right)^r\left(\frac{9}{10}\right)^{10,000-r}$

20. (*a*) $1-\left(\frac{1}{2}\right)^{221}$ (*b*) $1-\left(\frac{1}{2}\right)^{195}$

23. (*a*) $\frac{2}{9}$, (*b*) $10.\frac{5^3}{6^5}$, (*c*) $\frac{665}{729}$, (*d*) (*i*) $\frac{671}{1296}$, (*ii*) $1-\left(\frac{35}{36}\right)^{24}$, (*e*) $\frac{1}{2}\left(\frac{9}{10}\right)^4$, (*f*) $1-(0.95)^5$,

(*g*) $\sqrt{npq}$ (*h*) $np, \sqrt{npq}, \frac{q-p}{\sqrt{npq}}$ (*i*) $\frac{1}{343}$

35.13 POISSON DISTRIBUTION

Poisson distribution is a particular limiting form of the Binomial distribution when p (or q) is very small, n is indefinitely large and $np = m$ (say) is finite.

Probability function of X is given by

$$P(X = r) = \frac{m^r e^{-m}}{r\,!}$$

where m is the mean of the distribution.

Mean of Poisson Distribution

$$P(r) = \frac{e^{-m}\,.\,m^r}{r\,!}$$

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Successes r	Frequency f	$f.r$
0	$\dfrac{e^{-m}m^0}{0!}$	0
1	$\dfrac{e^{-m}m^1}{1!}$	$e^{-m}.\, m$
2	$\dfrac{e^{-m}m^2}{2!}$	$e^{-m}.\, m^2$
3	$\dfrac{e^{-m}m^3}{3!}$	$\dfrac{e^{-m}.\, m^3}{2!}$
…	…	…
r	$\dfrac{e^{-m}m^r}{r\,!}$	$\dfrac{e^{-m}.\, m^r}{(r-1)\,!}$
…	…	…

$$\sum f\ r = 0 + e^{-m}.\, m + e^{-m}.\, \frac{m^3}{2!} + \ldots + e^{-m}\, \frac{m^r}{(r-1)!} + \ldots \quad . \quad 1 \quad \ldots \quad \ldots \quad .$$

$$= e^{-m}.m\left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \ldots + \frac{m^{r-1}}{(r-1)!} + \ldots\right] = m\,.\, e^{-m}.[e^m] = m$$

$$\text{Mean} = \frac{\sum fr}{\sum f} = \frac{\text{m}}{1}, \quad \text{Mean} = m.$$ **Ans.**

Standard Deviation of Poisson Distribution S.D.= $\sqrt{m}$

∴ **Mean and variance of a Poisson distribution are each equal to *m*.**

$$\mu_3 = m,\ \mu_4 = 3m^2 + m$$

Hence the coefficients of skewness and Kurtosis are

$$\beta_1 = \frac{1}{m}, \qquad \beta_2 = 3 + \frac{1}{m}$$

$$\gamma_1 = \frac{1}{\sqrt{m}}, \qquad \gamma_2 = \frac{1}{m}$$

Example 1. *In a certain factory producing cycle tyres, there is a small chance of 1 in 500 tyres to be defective. The tyres are supplied in lots of 10. Using Poisson distribution, calculate the approximate number of lots containing no defective, one defective and two defective tyres, respectively, in a consignment of 10,000 lots.*

Solution. $$P = \frac{1}{500}, \quad n = 10$$

$$m = np = 10.\frac{1}{500} = \frac{1}{50} = 0.02, \qquad P(r) = \frac{e^{-m}.m^r}{r!}$$

S.No.	Probability of defective	Number of lots containing defective
1	$P(0) = \frac{e^{-0.02}(0.02)^0}{0!} = e^{-0.02} = 0.9802$	10,000 × 0.9802 = 9802 lots
2	$P(1) = \frac{e^{-0.02}(0.02)^1}{1!}$ $= 0.9802 \times 0.02 = 0.019604$	10,000 × 0.019604 = 196 lots
3	$P(2) = \frac{e^{-0.02}(0.02)^2}{2!}$ $= 0.9802 \times 0.0002 = 0.00019604$	10,000 × 0.000196 = 2 lots

Ans.

Example 2. *A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the number of days in a year on which*

(*i*) neither car is on demand (*ii*) a car demand is refused. ($e^{-1.5} = 0.2231$)

(*MDU Dec. 2010*)

Solution. $m = 1.5$

(*i*) If the car is not used, then demand (r) = 0

$$P(r) = \frac{e^{-m}.m^r}{r!}, \quad P(0) = \frac{e^{-1.5}(1.5)^0}{0!} = e^{-1.5} = 0.2231$$

Number of days in a year when the demand is zero = 365 × 0.2231 ≈ 81 **Ans.**

(*ii*) Some demand is refused if the number of demands is more than two *i.e.* $r > 2$.

$$P(r > 2) = P(3) + P(4) + ... = 1 - [P(0) + P(1) + P(2)]$$

$$= 1 - \left[\frac{e^{-1.5}(1.5)^0}{0!} + \frac{e^{-1.5}(1.5)^1}{1!} + \frac{e^{-1.5}(1.5)^2}{2!}\right]$$

$$= 1 - [e^{-1.5} + e^{-1.5} \times 1.5 + e^{-1.5} \times 1.125]$$

$$= 1 - e^{-1.5}[1 + 1.5 + 1.125] = 1 - e^{-1.5} \times 3.625$$

$$= 1 - 0.2231 \times 3.625 = 1 - 0.8087375$$

$$= 0.1912625$$

Number of days in a year when some demand of car is refused

$$= 365 \times 0.1912625 = 69.81 \approx 70 \text{ days}$$ **Ans.**

Example 3. *If the probability that an individual suffers a bad reaction from a certain injection is 0.001, determine the probability that out of2000 individuals*

(*a*) *exactly 3* (*b*) *more than 2 individuals* (*c*) *None* (*d*) *More than one individual will suffer a bad reaction.*

Solution. $p = 0.001, \quad n = 2000$

$$m = np = 2000 \times 0.001 = 2$$

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$\therefore \qquad P(r) = \frac{e^{-m}.m^r}{r!} = e^{-2}\frac{2^r}{r!} = \frac{1}{e^2}\times\frac{2^r}{r!}$

(*a*) $P\text{ (Exactly 3)} = P\,(3) = \frac{1}{e^2}.\frac{2^3}{3!} = \frac{1}{(2.718)^2}\times\frac{8}{6} = (0.135)\times\frac{4}{3} = 0.18$

(*b*) P (more than 2) = P (3) + P (4) + P (5) + ... + P (2000)

$= 1 - [P\,(0) + P\,(1) + P\,(2)]$

$= 1 - \left[\frac{e^{-2}(2)^0}{0!} + \frac{e^{-2}(2)^1}{1!} + \frac{e^{-2}(2)^2}{2!}\right]$

$= 1 - e^{-2}[1+2+2] = 1 - \frac{5}{e^2}$

$= 1 - 5 \times 0.135 = 1 - 0.675 = 0.325$ **Ans.**

(*c*) $P\text{ (none)} = P\,(0) = \frac{e^{-2}(2)^0}{0!} = 0.135$

(*d*) P (more than 1) = P (2) + P (3) + P (4) + ... + P (2000) = 1 – [P (0) + P (1)]

$= 1 - \left[\frac{e^{-2}(2)^0}{0!} + \frac{e^{-2}(2)^1}{1!}\right] = 1 - 3e^{-2} = 1 - 3\times 0.135 = 1 - 0.405 = 0.595$ **Ans.**

Example 4. *A manufacturer knows that the razor blades he makes contain on an average 0.5% of defectives. He packs them in packets of 5. What is the probability that a packet picked at random will contain 3 or more faulty blades ?*

Solution. $p = 0.5\% = 0.005,\ n = 5$

$m = np = 5 \times 0.005 = 0.025$

$p\,(r) = \frac{e^{-m}.\,m^r}{r!} = \frac{e^{-0.025}(0.025)^r}{r!}$

$P\text{ (3 or more)} = P\,(3) + P\,(4) + P\,(5) = \frac{e^{-0.025}(0.025)^3}{3!} + \frac{e^{-0.025}(0.025)4}{4!} + \frac{e^{-0.025}(0.025)^5}{5!}$

$= \frac{e^{-0.025}(0.025)^3}{3!}[20 + 5\,(0.025) + (0.025)^2]$

$= \frac{0.975 \times 0.000015625 \times 20.125625}{120}$

$= 0.000002555.$ **Ans.**

Example 5. *An insurance company found that only 0.01% of the population is involved in a certain type of accident each year. If its 1000 policy holders were randomly selected from the population, what is the probability that not more than two of its clients are involved in such an accident next year?* *(given that $e^{-0.1} = 0.9048$)*

(*D.U. Dec 2017*)

Solution. $p = 0.01\% = \frac{1}{100}\times\frac{1}{100} = \frac{1}{1000}, \quad n = 1000$

$$m = np = (1000) \times \frac{1}{1000} = \frac{1}{10} = 0.1$$

$$p(r) = \frac{e^{-m}\, m^r}{r!}$$

P (not more than 2) = P (0, 1 and 2) = P (0) + P (1) + P (2)

$$= \frac{e^{-0.1}(0.1)^0}{0!} + \frac{e^{-0.1}(0.1)^1}{1!} + \frac{e^{-0.1}(0.1)^2}{2!}$$

$$= e^{-0.1}\left(1 + 0.1 + \frac{0.01}{2}\right) = 0.9048 \times 1.105 = 0.9998$$

Ans.

EXERCISE 35.4

1. Find the probability that at most 5 defective fuses will be found in a box of 200 fuses if experience shows that 2 per cent of such fuses are defective.

2. The number of accidents during a year in a factory has the Poisson distribution with mean 1.5. The accidents during different years are assumed independent. Find the probability that only 2 accidents take place during 2 years time.

3. A manufacturer of cotter pins knows that 5% of his product is defective. If he sells cotter pins in boxes of 100 and guarantee that not more than 10 pins will be defective, what is the approximate probability that a box will fail to meet the guaranteed quality. [$e^{-5} = 0.006738$]

4. Suppose the number of telephone calls on an operator received form 9.00 to 9.05 follow a poisson distribution with mean 3. Find the probability that

(*i*) the operator will receive no calls in that time interval tomorrow,

(*ii*) in the next three days the operator will receive a total of 1 call in that time interval.

[$e^{-3} = 0.04978$]

5. On the basis of past record it has been found that there is a 70% chance of power-cut in a city on any particular day. What is the probability that from the first to the 10th day of the month, there are 5 or more days without power cut.

6. The distribution of typing mistakes committed by a typist is given below. Assuming a Poisson model, find out the expected frequencies:

Mistakes per pages	0	1	2	3	4	5
No. of pages	142	156	69	27	5	1

7. Let x be the number of cars per minute passing a certain crossing of roads between 5.00 P.M. and 7.00 P.M. on a holiday. Assume x has a Poisson distribution with mean 4. Find the probability of observing atmost 3 cars during any given minute between 5.00 P.M. and 7 P.M. (given $e^{-4} = 0.0183$)

8. Let x be the number of cars, passing a certain point, per minute at a particular time. Assuming that x has a poisson distribution with mean 0.5, find the probability of observing 3 or fewer cars during any given minute.

9. Number of customers arriving at a service counter during a day has a Poisson distribution with mean 100. Find the probability that at least one customer will arrive on each day during a period of five days. Also find the probability that exactly 3 customers will arrive during two days.

10. The random variable X has a Poisson distribution. If

$P(X = 1) = 0.01487$, $P(X = 2) = 0.04461$. Then find $P(X = 3)$.

11. A source of water is known to contain bacteria with mean number of bacteria per cc equal to 2. Five 1 cc test tubes were filled with water. Assuming that Poisson distribution is applicable, calculate the probability that exactly 2 test tubes contain at least 1 bacterium each.

12. In a normal summer, a truck driver gets on an average one puncture in 1000 km. Applying Poisson distribution, find the probability that he will have

(*i*) no puncture (*ii*) two punctures in a journey of 3000 kms.

13. Wireless sets are manufactured with 25 soldered joints each. On the average, 1 joint in 500 is defective. How many sets can be expected to be free from defective joints in a consignment of 10000 sets ?

14. In a certain factory turning out razor blades, there is small chance $\frac{1}{500}$ for any blade to be defective. The blades are supplied in packets of 10. Using Poisson's distribution, calculate the approximate number of packets containing (*i*) no defective (*ii*) one defective and (*iii*) two defective blades respectively in a consignment of 10,000 packets. ($e^{-0.02} = 0.9802$).

15. If m and μ_r denote by the mean and central rth moment of a Poisson distribution, then prove that

$$\mu_{r+1} = rm\mu_{r-1} + m\frac{d\mu_r}{dm}.\left[\text{Hint}.\ \mu_r = \sum_{n=0}^{\infty}(x-m)^r\frac{e^{-m}m^x}{x!},\ \text{find}\ \frac{d\mu_r}{dm}\right]$$

ANSWERS

1. 0.785 **2.** 0.224 **3.** 0.0136875

4. (i) e^{-3} (ii) $3 \times (e^{-3})^2(e^{-3}.\ 3)$ **5.** $\left(\frac{3^5}{5!}+\frac{3^6}{6!}+\frac{3^7}{7!}+\frac{3^8}{8!}+\frac{3^9}{9!}+\frac{3^{10}}{10!}\right)e^{-3}$

6. 147, 147, 74, 25, 6, 1 pages. **7.** 0.4331 **8.** 0.998

9. $(1 - e^{-100})^5,\ e^{-200} \times \frac{4(100)^3}{3}$ **10.** 0.08922

11. $\frac{2}{5}(1-e^{-2}) = 0.3459$ **12.** (*i*) e^{-3} (*ii*) $4.5\ e^{-3}$

13. 9512 **14.** (*i*) 9802 (*ii*) 196 (*iii*) 2

35.14 CONTINUOUS DISTRIBUTION

So far we have dealt with discrete distributions where the variate takes only the integral values. But the variates like temperature, heights and weights can take all values in a given interval. Such variables are called continuous variables.

Distribution function

If $F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)dx$, then $f(x)$ is defined as the Distribution Function.

Let $f(x)$ be a continuous function, then Mean $= \int_{-\infty}^{+\infty} xf(x)dx$

Variance $= \int_{-\infty}^{+\infty}(x-\bar{x})^2.f(x)dx.$ ($\bar{x}$ = mean)

Notes. $f(x)$ is called probability density function if

(1) $f(x) \ge$ G for every value of x. (2) $\int_{-\infty}^{\infty} f(x)dx = 1$. (3) $\int_a^b f(x)dx = P, (a < x < b)$.

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Example 1. *A function f (x) is defined as follows*

$$f(x)=\begin{cases}0, & x<2\\ \frac{1}{18}(2x+3), & 2\le x\le 4\\ 0, & x>4\end{cases}$$

Show that it is a probability density function.

Solution. $$f(x)=\begin{cases}0, & x<2\\ \frac{1}{18}(2x+3), & 2\le x\le 4\\ 0, & x>4\end{cases}$$

If f (x) is a probability density function, then

(*i*) $$\int_{-\infty}^{\infty} f(x)dx=1$$

Here $$\int_2^4 \frac{1}{18}(2x+3)dx=\frac{1}{18}\left[x^2+3x\right]_2^4=\frac{1}{18}(16+12-4-6)=1$$

(*ii*) $f(x)>0$ for $2\le x\le 4$

Hence, the given function is a probability density function.

Example 2. *The probability density function f (x) of a continuous random variable x is defined by* **(*Calcutta 2018, 2013*)**

$$f(x)=\begin{cases}\frac{A}{x^3}, & 5\le x\le 10\\ 0, & \text{otherwise}\end{cases}$$

Find the value of A.

Solution. Here, $$f(x)=\frac{A}{x^3},\qquad 5\le x\le 10$$

Since $f(x)$ is probability density function, so

$$\int_5^{10}\frac{A}{x^3}dx=1$$

$$\Rightarrow\quad \left[-\frac{A}{2x^2}\right]_5^{10}=1$$

$$\Rightarrow\quad \frac{A}{2}\left[-\frac{1}{100}+\frac{1}{25}\right]=1$$

$$\Rightarrow\quad \frac{A}{2}\left(\frac{3}{100}\right)=1\quad\Rightarrow\quad A=\frac{200}{3}$$ **Ans.**

Example 3. *The diameter of an electric cable is assumed to be continuous random variate with probability density function*:

f (x) = 6 x (l – x), 0 ≤ x ≤ 1

(*i*) *verify that above is a p.d.f* (*ii*) *find the mean and variance.*

Solution. (*i*) $\int_{-\infty}^{\infty} f(x)dx = \int_0^1 6x(1-x)dx = \int_0^1 (6x - 6x^2)dx$

$$= \left(3x^2 - 2x^3\right)_0^1 = 3 - 2 = 1$$

Secondly $f(x) > 0$ for $0 \le x \le 1$.

Hence the given function is a probability density function.

(*ii*) $\quad$ Mean $= \int_{-\infty}^{\infty} x.f(x)dx = \int_0^1 x.6x(1-x)dx$

$$= \int_0^1 (6x^2 - 6x^3)dx = \left(2x^3 - \frac{3}{2}x^4\right)_0^1 = 2 - \frac{3}{2} = \frac{1}{2}$$ **Ans.**

$$\text{Variance } = \int_{-\infty}^{\infty} (x - \overline{x})^2 . f(x)dx = \int_0^1 \left(x - \frac{1}{2}\right)^2 .6x(1-x)\, dx$$

$$= \int_0^1 \left(x^2 - x + \frac{1}{4}\right)(6x - 6x^2)dx = \int_0^1 \left(12x^3 - 6x^4 - \frac{15}{2}x^2 + \frac{3}{2}x\right)dx$$

$$= \left(3x^4 - \frac{6}{5}x^5 - \frac{5}{2}x^3 + \frac{3x^2}{4}\right)_0^1 = \left(3 - \frac{6}{5} - \frac{5}{2} + \frac{3}{4}\right) = \frac{1}{20}$$ **Ans.**

EXERCISE 35.5

1. The distribution function of a random variable X is given by

$$F(x) = \begin{cases} cx^3, & 0 \le x < 3 \\ 1, & x \ge 3 \\ 0, & x < 0 \end{cases}$$

If $P(X = 3) = 0$, Find (*a*) the constant c, (*b*) the density function, (*c*) $P(X > 1)$, (*d*) $P(1 < X < 2)$.

2. If a random variable X has density function

$$f(x) = \begin{cases} ce^{-3x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

Than calculate (*a*) the constant c, (*b*) $P(1 < X < 2)$, (*c*) $P(X \ge 3)$, (*d*) $P(X < 1)$

3. If a random variable X has density function

$$f(x) = \begin{cases} cx^2, & 1 \le x \le 2 \\ cx, & 2 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

Than calculate (*a*) the constant c, (*b*) $P(X > 2)$, (*c*) P $(1/2 < X < 3/2)$.

ANSWERS

1. (*a*) $\frac{1}{27}$, (*b*) $f(x) = \begin{cases} x^2/9, & 0 \le x < 3 \\ 0, & \text{otherwise} \end{cases}$, (*c*) $\frac{26}{27}$, (*d*) $\frac{7}{27}$

2. (*a*) 3 (*b*) $e^{-3} - e^{-6}$ (*c*) e^{-9} (*d*) $1 - e^{-3}$ $\quad$ **3.** (*a*) $\frac{6}{29}$, (*b*) $\frac{15}{29}$, (*c*) $\frac{19}{116}$

35.15 NORMAL DISTRIBUTION

Normal distribution is a continuous distribution. It is derived as the limiting form of the Binomial distribution for large values of n where neither p nor q is very small.

The normal distribution is given by the equation

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad ...(1)$$

where μ = mean, σ = standard deviation, π = 3.14159..., [e = 2.71828...]

$$P\,(x_1<x<x_2)=\int_{x_1}^{x_2}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$$

On substitution $z=\frac{x-\mu}{\sigma}$ in (1), we get $f(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ ...(2)

Here mean = 0, standard deviation = 1.

(2) is known as standard form of Normal Distribution.

Theorem. To derive Normal Distribution as a limiting case of Binomial distribution where $p \neq q$ but $p \approx q$. ***(U.P. III Semester Dec. 2006)***

Statement. The limiting case of Binomial Distribution $(p+q)^n$, as $n \to \infty$ and neither p nor. q are very small, generates the Normal Distribution.

Proof. The frequencies for r and $(r+1)$ successes in binomial distribution are

$f(r) = N.\,{}^nC_r\, p^r q^{n-r}$ and $f(r+1) = N.\,{}^nC_{r+1}\, p^{r+1} q^{n-(r+1)}$

The frequency of r successes > frequency of $(r+1)$ successes if

$$f(r)>f(r+1) \quad\Rightarrow\quad \frac{f(r)}{f(r+1)}>1$$

$$\Rightarrow \quad \frac{N.\,{}^nC_r\,P^rq^{n-r}}{N.\,{}^nC_{r+1}\,P^{r+1}q^{n-r-1}}>1 \quad\Rightarrow\quad \frac{\frac{n!}{r!(n-r)!}.P^r.q^{n-r}}{\frac{n!}{(r+1)!(n-r-1)!}P^{r+1}q^{n-r-1}}>1$$

$$\Rightarrow \quad \frac{n!\,P^r.q^{n-r}(r+1)!(n-r-1)!}{r!(n-r)!.n!\,P^{r+1}.q^{n-r-1}}>1$$

$$\Rightarrow \quad \frac{q.(r+1)}{(n-r)\,P}>1 \quad\Rightarrow\quad qr+q>np-pr$$

$$\Rightarrow \quad q > np - r\,(p+q)$$

$$\Rightarrow \quad r > np - q \qquad ...(1)$$

Again, similarly the frequency of r successes > the frequency of $(r-1)$ successes if

$$f(r)>f(r-1) \quad\Rightarrow\quad \frac{f(r)}{f(r-1)}>1$$

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$$\Rightarrow \quad \frac{N.\,{}^nC_r\,P^r q^{n-r}}{N.\,{}^nC_{r-1}\,P^{r-1} q^{n-(r-1)}} > 1 \quad \Rightarrow \quad \frac{\frac{n!}{r!(n-r)!}.P^r.q^{n-r}}{\frac{n!}{(r-1)!(n-r+1)!}P^{r-1}q^{n-r+1}} > 1$$

$$\Rightarrow \quad \frac{n!\,P^r.q^{n-r}(r-1)!(n-r+1)!}{r!(n-r)!.n!\,P^{r-1}.q^{n-r+1}} > 1 \quad \Rightarrow \quad \frac{P(n-r+1)}{r\,q} > 1$$

$$\Rightarrow \quad Pn - pr + p > rq \quad \Rightarrow \quad pn + p > pr + qr$$

$$\Rightarrow \quad pn + p > r(p+q) \quad \Rightarrow \quad pn + p > r \qquad \ldots(2)$$

$[\because\ p + q = 1]$

from (1) and (2), we have

$$pn + p > r > np - q$$

$$pn + p + q > r > np$$

$$np + 1 > r > np$$

Since a possible value of r is np, therefore, without loss of generality we can assume that np is an integer as $n \to \infty$. Hence the frequency of np successes can be assumed to be maximum frequency. Let $y0$ be the frequency of np successes and yx be the frequency of $(np + x)$ successes.

Then

$$y_0 = f(n\,p) = N\,.\,{}^nC_{n\,p}\,p^{n\,p}\,q^{n-n\,p} \qquad \text{[from (1), for } r = n\,p\text{]}$$

$$= N\frac{n!}{(n\,p)!\,(n\,q)!}p^{np}q^{n-np}$$

$$= N\frac{n!}{(n\,p)!\,(n\,q)!}p^{np}q^{nq} \qquad \ldots(3) \quad [\because\ q = 1 - p]$$

and

$$y_x = N.\frac{n!}{(n\,p+x)!\,(n\,q-x)!}p^{n\,p+x}q^{nq-x} \qquad \ldots(4)$$

Dividing (4) by (3), we get

$$\frac{y_x}{y_0} = N.\frac{(n\,p)!\,(n\,q)!}{(n\,p+x)!\,(n\,q-x)!}p^x q^{-x} \qquad \ldots(5)$$

For n being large, then according to James Stirling's approximation formula for factorials, we have

$$n! = e^{-n}\,h^{n+1/2}\sqrt{(2\pi)},$$

From (5) $$\frac{y_x}{y_0} = \frac{e^{-np}(n\,p)^{n\,p+1/2}\sqrt{2\pi}\,e^{-nq}(nq)^{nq+1/2}\sqrt{2\pi}\;p^x q^{-x}}{e^{-(n\,p+x)}(n\,p+x)^{n\,p+x+1/2}\sqrt{2\pi}\,e^{-(nq-x)}(n\,q-x)^{n\,q-x+1/2}\sqrt{2\pi}}$$

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$$= \frac{(np)^{n\,p+1/2}\,(nq)^{nq+1/2}\,(nq/nq)^{x}}{(np)^{n\,p+x+1/2}\left\{1+\dfrac{x}{n\,p}\right\}^{n\,p+x+1/2}(np)^{nq-x+1/2}\left\{1-\dfrac{x}{n\,q}\right\}^{nq-x+1/2}}$$

$$= \frac{1}{\left\{1+\dfrac{x}{n\,p}\right\}^{n\,p+x+1/2}\left\{1-\dfrac{x}{n\,q}\right\}^{nq-x+1/2}}$$

$$\therefore \quad \log\left(\frac{y_x}{y_0}\right) = -\left(np+x+\frac{2}{2}\right)\log\left(1+\frac{x}{np}\right)-\left(nq-x+\frac{1}{2}\right)\log\left(1-\frac{x}{nq}\right)$$

$$= -\left(np+x+\frac{1}{2}\right)\left(\frac{x}{nq}-\frac{x^2}{2n^2p^2}+\frac{x^3}{3n^3q^3}-....\right)$$

$$+\left(nq-x+\frac{1}{2}\right)\left(\frac{x}{nq}+\frac{x^2}{2n^2q^2}+\frac{x^3}{3n^3q^3}-....\right)$$

$$= x\left(1-\frac{1}{2np}+1+\frac{1}{2np}\right)+x^2\left(\frac{1}{2np}-\frac{1}{np}+\frac{1}{4n^2p^2}+\frac{1}{2nq}-\frac{1}{nq}+\frac{1}{4n^2q^2}\right)$$

$$+x^3\left(\frac{1}{3n^2q^2}+\frac{1}{6n^3q^3}-\frac{1}{2n^2q^2}-\frac{1}{2n^2p^2}-\frac{1}{3n^2q^2}+\frac{1}{6n^3p^3}\right)-....$$

$$= \frac{p-q}{2npq}x+\frac{p^2+q^2}{4n^2p^2q^2}x^2-\frac{x^2}{2npq}+....\ +\text{terms of higher orders}.$$

Neglecting terms containing $1/n^2$, we have

$$\therefore \quad \log\left(\frac{y_x}{y_0}\right) = -\frac{q-p}{2npq}x-\frac{x^2}{2npq}$$

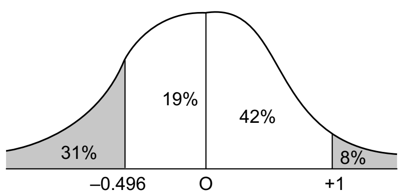
Since $p < 1$, $q < 1$ and so $q - p$ is very small as compared with n. Therefore Ist term may be neglected, $(q - p = 0)$.

$$\therefore \quad \log\left(\frac{y_x}{y_0}\right) = -\frac{x^2}{2npq} = -\frac{x^2}{2\sigma^2} \quad [\because\ \sigma^2 = npq, \text{ the variance of Binomial distribution}]$$

$$\Rightarrow \quad y_x = y_0\, e^{-x^2/2\sigma^2}$$ **Proved.**

Example 1. *In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and standard deviation of the distribution.* **(*D.U. 2016*)**

Solution. Let $\bar{x}$ be the mean and σ the S.D.



If $x = 45$, $\quad z = \dfrac{45-\bar{x}}{\sigma}$

If $x = 64$, $\quad z = \dfrac{64-x}{\sigma}$

Area between 0 and $\quad z = \dfrac{45-\bar{x}}{\sigma} = 0.50 - 0.31 = 0.19$

[From the table, for the area 0.19, $z = 0.496$]

$$\frac{45 - \bar{x}}{\sigma} = -0.496 \qquad ...(1)$$

Area between $z = 0$ and $z = \frac{64 - \bar{x}}{\sigma} = 0.5 - 0.08 = 0.42$.

(From the table, for area 0.42, $z = 1.405$)

$$\frac{64 - \mu}{\sigma} = 1.405 \qquad ...(2)$$

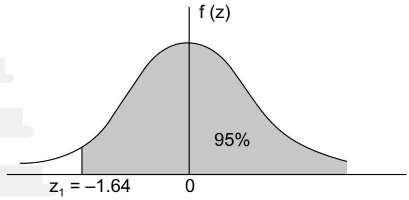
Solving (1) and (2) we get $\mu = 50$, $\sigma = 10$. **Ans.**

Example 2. *The income of a group of 10,000 persons was found to be normally distributed with mean ₹. 750 p.m. and standard deviation of ₹. 50. Show that, of this group, about 95% had income exceeding ₹. 668 and only 5% had income exceeding ₹. 832. Also find the lowest income among the richest 100.*

Solution. Mean $= \mu = 750$

Standard deviation $= \sigma = 50$

and $z = \frac{x - \mu}{\sigma}$

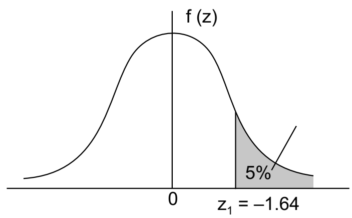


(*i*) If $x_1 = 668$, then $z_1 = \frac{668 - 750}{50} = -1.64$

$$\begin{aligned} P(x_1 > 668) &= P(z_1 < -1.64) \\ &= 0.5 + P(-1.64 \le z \le 0) \\ &= 0.5 + P(0 \le z \le 1.64) \\ &= 0.5 + 0.4495 \\ &= 0.9495 \end{aligned}$$

∴ Percentage of persons having income exceeding ₹. 668 = 94.95% ≈ 95% (approx.)

(*ii*) If $x = 832$, then $z = \frac{832 - 750}{50} = 1.64$



$$\begin{aligned} P(x_2 > 832) &= P(z_2 > 1.64) \\ &= 0.5 - 0.4495 \\ &= 0.0505 \end{aligned}$$

∴ Percentage of persons having income exceeding ₹. 832 = 5.05% = 5% (approx.)

(*iii*) Let x be the lowest income among the richest 100 persons.

100 persons =1% of 10,000

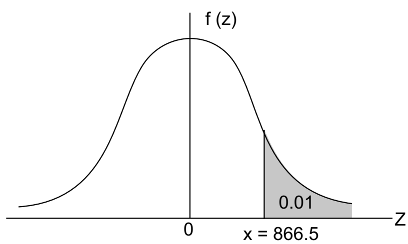
100 persons represents 1% area under the curve on the right hand side.

Thus the area between 0 and z

$= 0.5 - 0.01 = 0.49$

From the table z for area 0.49 is 2.33

$$z = \frac{x - \mu}{\sigma}$$



$\Rightarrow \qquad 2.33 = \dfrac{x - 750}{50} \quad \Rightarrow \quad x - 750 = 50 \times 2.33$

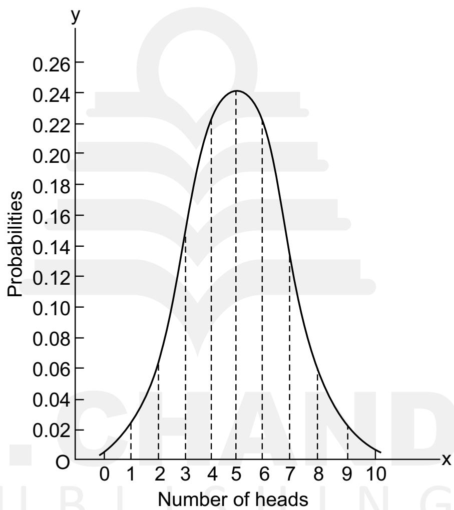
$\Rightarrow \qquad x - 750 = 116.5 \quad \Rightarrow \quad x = 866.5$

Hence, the minimum income among the 100 richest persons is equal to ₹. 866.5. **Ans.**

35.16 NORMAL CURVE

A Normal Curve shows binomial distribution graphically of a continuous Random Variable. The probabilities of heads in 10 tosses are ,${}^{10}C_0\, q^{10}\, p^0$, ${}^{10}C_1\, q^9\, p^1$, ${}^{10}C_2\, q^8\, p^2$, ${}^{10}C_3\, q^7\, p^3$, ${}^{10}C_4\, q^6\, p^4$, ${}^{10}C_5\, q^5\, p^5$, ${}^{10}C_6\, q^4\, p^6$, ${}^{10}C_7\, q^3\, p^7$, ${}^{10}C_8\, q^2\, p^8$, ${}^{10}C_9\, q^1\, p^9$, ${}^{10}C_{10}\, q^0\, p^{10}$.

$p = \dfrac{1}{2}, q = \dfrac{1}{2}$. It is shown in the figure given below.



If the variates (heads here) are treated as if they were continuous, the required probability curve will be a *normal curve* as shown in the above figure by dotted line.

Properties of the normal curve, $y = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

(1) The curve is symmetrical about the line. $x = \mu$.

(2) The mean, median and mode coincide.

(3) y decreases rapidly as x increases numerically. The curve extends to infinity on either side of the origin.

(4) (*a*) $P\,(\mu - \sigma < x < \mu + \sigma) = 0.6826$

(*b*) $P\,(\mu - 2\,\sigma < x < \mu + 2\,\sigma) = 0.9544$

(*c*) $P\,(\mu - 3\,\sigma < x < \mu + 3\,\sigma) = 0.9973$

Hence (*a*) About 68% of the values lie between $(\mu - \sigma)$ and $\mu + \sigma$

(*b*) About 95% of the values lie between $(\mu - 2s)$ and $(\mu + 2s)$.

(*c*) About 99.7 % of the values will be between $(\mu - 3\,\sigma)$ and $(\mu + 3\sigma)$.

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(5) $\beta_1 = 0$ and $\beta_2 = 3$.

(6) x-axis is an asymptote to the curve. No portion of the curve lies below the x-axis.

(7) The points of inflexion are $x = \mu \pm s$.

(8) Mean deviation about mean $\simeq \frac{4}{5}\sigma$ and quartile deviation $\simeq \frac{2}{3}\sigma$.

35.17 MEDIAN OF THE NORMAL DISTRIBUTION

If a is the median, then it divides the total area into two equal halves so that

(***Vidyasagar University 2018***)

$$\int_{-\infty}^{a} f(x)dx = \frac{1}{2} = \int_{a}^{\infty} f(x)dx$$

where $$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Suppose Median $a >$ mean μ then

$$\int_{-\infty}^{\mu} f(x)dx + \int_{\mu}^{a} f(x)dx = \frac{1}{2} \qquad \left[\text{but } \int_{-\infty}^{\mu} f(x)dx = \frac{1}{2}\right]$$

$$\frac{1}{2} + \int_{\mu}^{a} f(x)dx = \frac{1}{2} \qquad (\mu = \text{mean})$$

$$\int_{\mu}^{a} f(x)dx = 0$$

Thus $$a = \mu$$

Similarly, when $a <$ mean, we have $a = \mu$.

Thus, median = mean = μ.

Mode of the normal distribution

We know that mode is the value of the variate x for which $f(x)$ is maximum. Thus, by differential 1 calculus $f(x)$ is maximum if $f(x) = 0$ and $f ¢ (x) < 0$

where $$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Clearly $f(x)$ will be maximum when the exponent will be maximum which will be the case

$$\frac{(x-\mu)}{2\sigma^2} = 0 \;\Rightarrow\; (x-\mu)^2 = 0 \;\Rightarrow\; x - \mu$$

Thus mode is μ, and modal ordinate $= \frac{1}{\sigma\sqrt{2\pi}}$

35.18 AREA UNDER THE NORMAL CURVE

By taking $z = \dfrac{x - \overline{x}}{\sigma}$, the standard normal curve is formed.

The total area under this curve is 1. The area under the curve is divided into two equal parts by $z = 0$. Left hand side area and right hand side area to $z = 0$ is 0.5. The area between the ordinate $z = 0$ and any other ordinate can be noted from the table:- 1 on last page 34 of the chapter.

Example 1. *In mathematics final examination, if the mean was 72, and the standard deviation was 15. Determine the standard scores of students receiving grades*:

(*a*) 60 (*b*) 93 (*c*) 72

Solution. Here, $\overline{x} = 72$, $\sigma = 15$

(a) $z = \dfrac{x - \overline{x}}{\sigma} = \dfrac{60 - 72}{15} = -0.8$ (b) $z = \dfrac{93 - 72}{15} = 1.4$ (c) $z = \dfrac{72 - 72}{15} = 0$ **Ans.**

Example 2. *Find the area under the normal curve in each of the cases*

(*a*) *$z = 0$ and $z = 1.2$;* (*b*) *$z = -0.68$ and $z = 0$;*

(c) *$z = -0.46$ and $z = 2.21$;* (*c*) *$z = 0.81$ and $z = 1.94$;*

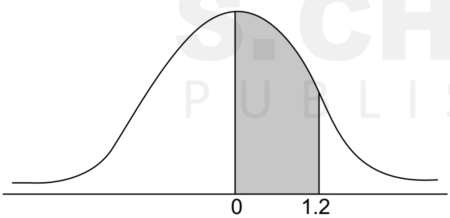
(*e*) *To the left of $z = 0.6$;* (*f*) *Right of $z = -1.28$.*

Solution.

See table –1, last page of the chapter.

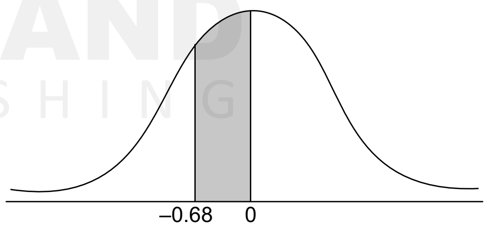
(*a*) Area between $z = 0$ and $z = 1.2$

$= .3849$ **Ans.**

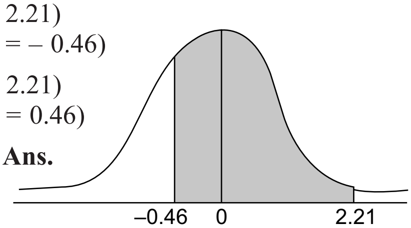


(*b*) Area between $z = 0$ and $z = -0.68$

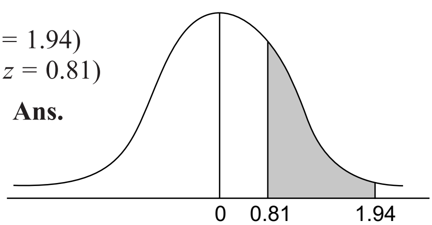
$= 0.2518$ **Ans.**



(*c*) Required area = (Area between $z = 0$ and $z = 2.21$)
+ (Area between $z = 0$ and $z = -0.46$)
= (Area between $z = 0$ and $z = 2.21$)
+ (Area between $z = 0$ and $z = 0.46$)
= 0.4865 + 0.1772 = 0.6637. **Ans.**



(*d*) Required area = (Area between $z = 0$ and $z = 1.94$)
– (Area between $z - 0$ and $z = 0.81$)
= 0.4738 – 0.2910 = 0.1828 **Ans.**



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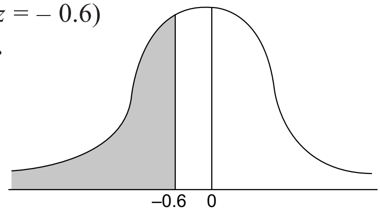
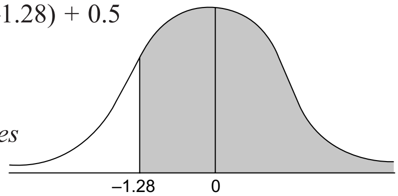
(e) Required area = 0.5 – (Area between z = 0 and z = – 0.6)
= 0.5 – 0.2257 = 0.2743. **Ans.**



(f) Required area = (Area between z = 0 and z = –1.28) + 0.5
= 0.3997 + 0.5
= 0.8997. **Ans.**



Example 3. *Find the value of z in each of the cases*

(a) *Area between 0 and z is 03770*

(b) *Area to the left of z is 0.8621*

Solution.

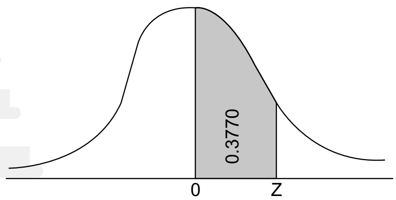
(a) z = ± 1.16

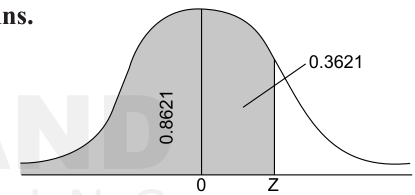
(b) Since the area is greater than 0.5.

Area between 0 and z.

= 0.8621 – 0.5 = 0.3621

from the table –1 z = 1 + 0.09 = 1.09 **Ans.**

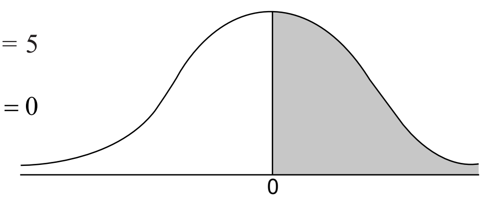




Example 4. *Students of a class were given an aptitude test Their marks were found to be normally distributed with mean 60 and standard deviation 5. What percentage of students scored more than 60 marks ?*

Solution. $x = 60,\ \bar{x} = 60,\ \sigma = 5$

$$z = \frac{x-\bar{x}}{\sigma} = \frac{60-60}{5} = 0$$



if $x > 60$ then $z > 0$

Area lying to the right of z = 0 is 0.5.

The percentage of students getting more than 60 marks = 50 %

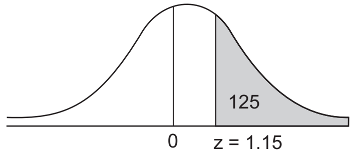
Example 5. *Assume mean height of soldiers to be 68.22 inches with a variance of 1.0.8 inches square. How many soldiers in a regiment of 1,000 would you expect to be over 6 feet tall, given that the area under the standard normal curve between x = 0 and x = 0.35 is 0.1368 and between x = 0 and x = 1.15 is 0.3746.*

Solution. Mean = $\bar{x}$ = 68.22 inch

variance = σ^2 = 10.8 inches squares

If $x = 72$ inches then $\quad z = \frac{x - \overline{x}}{\sigma} = \frac{72 - 68.22}{\sqrt{10.8}} = 1.15$

$$P(x > 72) = P(z > 1.15)$$
$$= 0.5 - P(0 \le z \le 1.15)$$
$$= 0.5 - 0.3746 = 0.1254$$



Number of soldiers = 1000 × 0.1254 = 125.4 = 125 (app.) **Ans.**

Area right to $z = 2$ is 0.5 – 0.4772 = 0.0228

Number of workers getting ₹ more than 160 = 0.0228 × 1000 = 22.8 ≈ 23 **Ans.**

EXERCISE 35.6

1. In a regiment of 1000, the mean height of the soldiers is 68.12 units and the standard deviation is 3.374 units. Assuming a normal distribution, how many soldiers could be expected to be more than 72 units? It is given that

$P(z = 1.00) = 0.3413$, $P(z = 1.15) = 0.3749$ and

$P(z = 1.25) = 0.3944$, where z is the standard normal variable.

2. The lifetime of radio rubes manufactured in a factory is known to have an average value of 10 years. Find the probability that the lifetime of a tube taken randomly (*i*) exceeds 15 years, (*ii*) is less than 5 years, assuming that the exponential probability law is followed.

3. Analysis of past data shows that hub thickness of a particular type of gear is normally distributed about a main thickness of 2.00 cm with a standard deviation of 0.04 cm.

(*i*) What is the probability that a gear chosen at random will have a thickness greater than 2.06 cm?

(*ii*) How many gears in a production run of 600 such gears will have a thickness between 1.89 and 1.95 cm? Given $\phi(1.5) = 0.4332$, $\phi(2.75) = 0.4970$, $\phi(1.25) = 0.3944$

4. The breaking strength X of a cotton fabric is normally distributed with $E(x) = 16$ and s $(x) = 1$. The fabric is said to be good if $X \ge 14$. What is the probability that a fabric chosen at random is good. Given that $\phi(2) = 0.9772$

5. A manufacturer knows from experience that resistance of resistors he produces is normal with mean m =140 W and standard deviation $\sigma = 5\Omega$. Find the percentage of resistors that will have resistance between 138 Ω and 142 Ω. (given $\phi(0.4) = 0.6554$, where z is standard normal variate).

6. A manufacturing company packs pencils in fancy plastic boxes. The length of the pencils is normally distributed with μ = 6" and σ = 0.2" The internal length of the boxes is 6.4". What is the probability that the box would be too small for the pencils? (Given that a value of the standardized normal distribution function is $\phi(2) = 0.9772$).

7. A manufacturer produces airmail envelopes, whose weight is normal with mean μ – 1.95 gm and standard deviation σ = 0.05 gm. The envelopes are sold in lots of 1000. How many envelopes in a lot will be heavier than 2 gm? Use the fact that $\frac{1}{\sqrt{2\pi}}\int_2^1 \exp\left(\frac{-x^2}{2}\right)dx = 0.3413$

8. The mean height of 500 students is 151 cm and the standard deviation is 15 cm. Assuming that the heights are normally distributed, find how many students, height lie between 120 and 155 cm.

9. A large number of measurements is normally distributed with a mean of 65.5″ and S.D. of 6.2″. Find the percentage of measurements that fall between 54.8″ and 68.8″.

10. Find the mean and variance of the density function $f(x) = \lambda e^{-\lambda x}$

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11. If x is normally distributed with mean 1 and variance 4,

(*i*) Find $Pr\,(-3 \le x \le 3)$; (*ii*) Obtain k if $Pr\,(x \le k) = 0.90$

12. A normal variable x has mean 1 and variance 4. Find the probability that $x \ge 3$. (Given: z is the standard normal variable and $\phi\,(0) = 0.5$, $\phi\,(0.5) = 0.6915$, $\phi\,(1) = 0.8413$, $\phi\,(1.5) = 0.9332$)

13. (*a*) If x is normally distributed with mean 4 and variance 9; find

(*i*) $Pr\,(2.55 \le x \le 5.5)$. (*ii*) Obtain k if $Pr\,(x \le k) = 0.9$.

Use $Pr\,(z \le .5) = 0.691$ and $Pr\,(z \le 1.3) = 0.90$.

(*b*) If $\log_e x$ is normally distributed with mean 1 and variance 4, find $\left(\frac{1}{2} < x < 2\right)$ given that $\log_e(2) = 0.693$.

(*c*) For a standard normal variate z $P\,(-0.72 \le z \le 0) =$.......

14. The random variable x is normally distributed with $E\,(x) = 2$ and variance $V\,(x) = 4$. Find a number p (approximately), such that $P\,(x > p) = 2P\,(x \le p)$. [The values of the standard normal distribution are $\phi\,(-0.43) = 0.3336$, and $\phi\,(-0.44) = 0.3300$].

If $X \sim N\,(10, 4)$ find $Pr\,[|X| \ge 5]$.

15. The continuous random variable x is normally distributed with $E\,(x) = \mu$ and $V\,(x) = \mu^2$. If $Y = cx + d$, then find $V\,(Y)$.

16. The pdf of X is given by $f\,(X) = \lambda e^{-\lambda x}$ $x \ge 0$, λ 0. Calculate $Pr\,[X > E\,(X)]$.

If $X \sim N\,(75, 25)$, find $Pr\,[X > 80/X > 77]$

If $X \sim N\,(10, 4)$ find $Pr\,[|X| \ge 5\}$

17. A random variable x has a standard normal distribution ϕ. Prove : $Pr\,(1|X| > k) = 2[1 - \phi\,(k)]$

18. The random variable x has the probability density function $f\,(X) = kx$ if $0 \le x \le 2$

Find k. Find x such that (*i*) $Pr\,(X \le x) = 0.1$ (*ii*) $Pr\,(X \le x) = 0.95$

19. For a normal curve, show that $\mu_{2n+1} = 0$ and $\mu_{2n} = (2n - 1)\,\sigma^2\,\mu_{2n-2}$.

20. The length of an item manufactured on an automatic machine tool is a normally distributed random variable with parameters M [$\bar{x}$] =10 and $\mu^2 = \frac{1}{200}$. Find the probability of defective production if the balance is 10 + 0.05.

21. In a mathematics examination, the average grade was 82 and the standard deviation was 5. All the students with grades from 88 to 94 received a grade B. If the grades are normally distributed and 8 students received a B grade, find how many students took the examination. Given

$\frac{x}{\sigma}$	1.20	2.00	2.40	2.45
A	0.3849	0.4772	0.4918	0.4929

22. Explain the characteristics and importance of a normal distribution.

23. The life time of a certain component has a mean life of 400 hours and standard deviation of 50 hours. Assuming normal distribution for the life time of 1000 components, determine approximately the number of components whose life time lies between 340 to 465 hours. You may use the. Following data Where symbols have their usual meanings.

24. For standard normal variate mean μ is

(*a*) 1 (*b*) 0 (*c*) 6 (*d*) none of the above

25. Fill in the blanks:

(*a*) The mean of the marks obtained by the students is 50 and the variance is 25. If a student gets 60 marks, his standard score is ....

(*b*) If $(x) = \left(\frac{1}{\sqrt{2\pi}}\right)e^{\frac{-x^2}{2}}$, then its mean is .... and standard deviation is ....

(*c*) In the standard normal curve the area between $z = -1$ and $z = 1$ is nearly ....

(*d*) If $\sigma = 2$, $\bar{x} = 5$, the equation of normal distribution is ....

(*e*) The marks obtained were found normally distributed with mean 75 and variance 100. The percentage of students who scored more than 75 marks is ....

(*f*) The mean, median and mode of a normal distribution are ....

(*g*) Exponential distribution $f(x)$ is defined by $f(x) = ae^{-2x}$, $0 < x < \infty$, then $a =$ ....

(*h*) The probability density function of Beta distribution with a = 1, b = 4 is $f(x) =$ ....

(*i*) For a standard normal variate z $P(-0.72 \le z \le 0) =$ ....

ANSWERS

1. 125

2. (*i*) 0.2231, (*ii*) 0.3935.

3. (*i*) 0.0668, (*ii*) 62 (61.56) app.

4. 0.9772

5. 31.08%

6. 0 0228.

7. 159

8. 294

9. 66.01%

10. $\frac{1}{\lambda}, \frac{1}{\lambda^2}$

11. (*i*) 0.8185, (*ii*) 3.56

12. 0.1587

13. (*a*) (*i*) 0.382, (*ii*) 7.9. (*b*) 0.24. (*c*) 0.2642

14. 1.13834, $\frac{1}{e}, \frac{1}{5\sqrt{2\pi}}e^{-\frac{(x-75)^2}{2(0.5)}}$, 0.062

15. $c^2\ \mu^2$

16. $\frac{1}{e}, \frac{1}{5\sqrt{2\pi}}e^{-\frac{(x-75)^2}{2(0.5)}}$, 0.062

18. $k = \frac{1}{2}$, (*i*) $x = 0.632$, (*ii*) $x = 1.949$

20. 0.4798

21. 75 students

23. 788

24. (*b*)

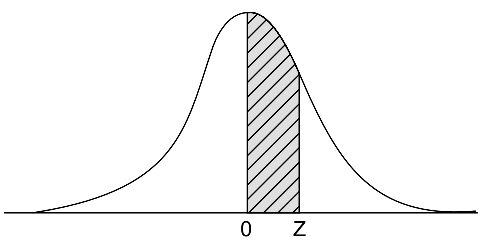
25. (*a*) 2, (*b*) 0,1, (*c*) 68%, (d) $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(x-5)^2}{8}}$, (*e*) 50%, (*f*) zero, (*g*) 2, (*h*) $4(1-x)^3$, (*i*) 0.2642

TABLE – 1

AREA UNDER STANDARD NORMAL CURVE FROM Z = 0 TO $Z = \frac{x - \bar{x}}{\sigma}$

An entry in the table is the proportion under the entire curve which is between Z = 0 and a positive value of Z. Area for negative values of Z are obtained by symmetry.

For different values of Z, table gives area (shown shaded in the figure) under normal curve.



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↓ Z→	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2703	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131.	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	: .4515	.4525	.4535	.4545
1.7	.4452	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931’	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4987	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4940	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

35.19 HYPOTHESIS TESTING

On the basis of sample information, we make certain decisions about the population. In taking such decisions we make certain assumptions. These assumptions are known as *statistical hypothesis*. These hypothesis are tested. Assuming the hypothesis is correct we calculate the probability of getting the observed sample. If this probability is less than a certain assigned value, the hypothesis is to be rejected.

Null Hypothesis (H_0)

Null hypothesis is based for analysing the problem. Null hypothesis is the *hypothesis of no difference*. Thus, we shall presume that there is no significant difference between the observed value and expected value. Then, we shall test whether this hypothesis is

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satisfied by the data or not. If the hypothesis is not approved the difference is considered to be significant. If hypothesis is approved then the difference would be described as due to sampling fluctuation. Null hypothesis is denoted by H_0.

Errors

In sampling theory to draw valid inferences about the population parameter on the basis of the sample results.

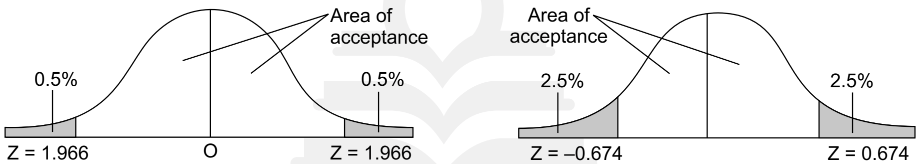
We decide to accept or to reject the lot after examining a sample from it. As such, we are liable to commit the following two types of errors.

Type 1 Error. If H_0 is rejected while it should have been accepted.

Type II Error. If H_0 is accepted while it should have been rejected.

Level of Significance

There are two critical regions which cover 5% and 1% areas of the normal curve. The shaded portions are the critical regions.



Thus, the probability of the value of the variate falling in the critical region is the level of significance. If the variate falls in the critical area, the hypothesis is to be rejected.

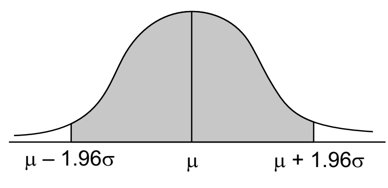
Test of Significance

The tests which enables us to decide whether to accept or to reject the null hypothesis is called the tests of significance. If the difference between the sample values and the population values are so large (lies in critical area), it is to be rejected.

Confidence Limits

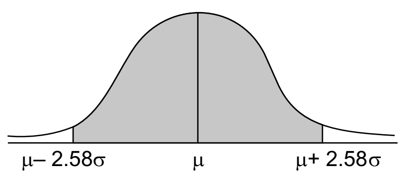
$\mu - 1.96\sigma$, $\mu + 1.96\sigma$ are 95% confidence limits as the area between $\mu - 1.96\sigma$ and $\mu + 1.96\sigma$ is 95%. If a sample statistics lies in the interval $\mu - 1.96\sigma$. $\mu + 1.96\sigma$, we call 95% confidence interval.

Similarly, $\mu - 2.58\sigma$, $\mu + 2.58\sigma$ is 99% confidence limits as the area between $\mu - 2.58\sigma$ and $\mu + 2.58\sigma$ is 99%. The numbers 1.96, 2.58 are called confidence coefficients.



Test of Significance of Large Samples ($n > 30$)

Normal distribution is the limiting case of Binomial distribution when n is large enough. For normal distribution 5% of the items lie outside $\mu \pm 1.96\ \sigma$ while only 1% of the items outside $\mu \pm 2.586\ \sigma$.



$$z = \frac{x - \mu}{\sigma}$$

where z is the standard normal variate and x is the observed number of successes.

First we find the value of z. Test of significance depends upon the value of z.

(i) (a) If $|z| < 1.96$, difference between the observed and expected number of successes is not significant at the 5% level of significance.

(b) If $|z| > 1.96$, difference is significant at 5% level of significance.

(ii) (a) If $|z| < 2.58$, difference between the observed and expected number of successes is not significant at 1% level of significance.

(b) If $|z| > 2.58$, difference is significant at 1% level of significance.

Example 1. *A cubical die was thrown 9,000 times and 1 or 6 was obtained 3120 times. Can the deviation from expected value lie due to fluctuations of sampling?*

Solution. Let us consider the hypothesis that the die is an unbiased one and hence the probability of obtaining 1 or 6 = $\frac{2}{6} = \frac{1}{3}$ *i.e.*, $p = \frac{1}{3},\ q = \frac{2}{3}$

The expected value of the number of successes = $np = 9000 \times \frac{1}{3} = 3000$

Also $\qquad \sigma = \text{S.D.} = \sqrt{npq} = \sqrt{9000 \times \frac{1}{3} \times \frac{2}{3}} = \sqrt{2000} = 44.72$

$3\sigma = 3 \times 44.72 = 134.16$

Actual number of successes = 3120

Difference between the actual number of successes and expected number of successes = 3120 – 3000 = 120 which is $< 3\sigma$

Hence, the hypothesis is correct and the deviation is due to fluctuations of sampling due randon causes. **Ans.**

Sampling Distribution of the Proportion

A simple sample of n items is drawn from the population. It is same as a series of n independent trials with the probability P of success. The probabilities of 0, 1, 2, …., n success are the terms in the binomial expansion of $(q + p)^n$.

Here mean = np and standard deviation = $\sqrt{npq}$.

Let us consider the proportion of successes, then

(a) Mean proportion of successes = $\frac{np}{n} = p$

(b) Standard deviation (standard error) of proportion of successes = $\frac{\sqrt{npq}}{n} = \sqrt{\frac{pq}{n}}$

(c) Precision of the proportion of success = $\frac{1}{\text{S.D.}} = \sqrt{\frac{n}{pq}}$.

Example 2. *A group of scientist reported 1705 sons and 1527 daughters. Do these figures conform to the hypothesis that the sex ratio is* $\frac{1}{2}$.

Solution. The total number of observations = 1705 + 1527 = 3232

The number of sons = 1705

Therefore, the observed male ratio = $\frac{1705}{3232} = 0.5175$

On the given hypothesis the male ratio = 0.5000

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Thus, the difference between the observed ratio and theoretical ratio = 0.5275 – 0.5000 = 0.0275

The standard deviation of the proportion = $\sqrt{\dfrac{pq}{n}} = \sqrt{\dfrac{\frac{1}{2}\times\frac{1}{2}}{3232}} = 0.0088$

The difference is more than 3 times of standard deviation.

Hence, it can be definitely said that the figures given do not conform to the given hypothesis.

Estimation of the Parameters of the Population

The mean, standard deviation etc. of the population are known as parameters. They are denoted by μ and σ. Their estimates are based on the sample values. The mean and standard deviation of a sample are denoted by $\bar{x}$ and s respectively. Thus, a statistic is an estimate of the parameter. There are two types of estimates.

(*a*) ***Point estimation:*** An estimate of a population parameter given by a single number is called a point estimation of the parameter. For example,

$$(\text{S.D.})^2 = \frac{\sum(x-\bar{x})^2}{n-1}$$

(*b*) ***Interval estimation:*** An interval in which population parameter may be expected to lie with a given degree of confidence. The intervals are

(*i*) $\bar{x} - \sigma_s$ to $\bar{x} + \sigma_s$ (68.27% confidence level)

(*ii*) $\bar{x} - 2\,\sigma_s$ to $\bar{x} + \sigma_s$ (95.45% confidence level)

(*iii*) $\bar{x} - 3\,\sigma_s$ to $\bar{x} + 3\,\sigma_s$ (99.13% confidence level)

$\bar{x}$ and σ_s are the mean and S.D. of the sample.

Similarly, $\bar{x} \pm 1.96\,\sigma_s$, $\bar{x} \pm 2.58\,\sigma_s$ are 95% and 99% confidence of limits for μ.

$$\bar{x} \pm 1.96\,\frac{\sigma}{\sqrt{n}} \text{ and } \bar{x} \pm 2.58\,\frac{\sigma}{\sqrt{n}} \text{ are also the intervals as } \sigma_s = \frac{\sigma}{\sqrt{n}}.$$

Comparison of Large Samples

Let two large samples of size n_1, n_2 be drawn from two populations of proportions of attributes A's as p_1, p_2 respectively.

(*i*) ***Hypothesis:*** As regards the attribute A, the two populations are similar. On combining the two samples

$$p = \frac{n_1\,p_1 + n_2\,p_2}{n_1 + n_2}$$

where p is the common proportion of attributes.

Let e_1, e_2 be the standard errors in the two samples, then

$$e_1^2 = \frac{pq}{n_1} \text{ and } e_2^2 = \frac{pq}{n_2}$$

If e be the standard error of the combined samples, then

$$e^2 = e_1^2 + e_2^2 = \frac{pq}{n_1} + \frac{pq}{n_2} = pq\left[\frac{1}{n_1} + \frac{1}{n_2}\right]$$

$$Z = \frac{p_1 - p_2}{e}$$

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1. If $Z > 3$, the difference between p_1 and p_2 is significant.
2. If $Z < 2$, the difference may be due to fluctuations of sampling.
3. If $2 < Z < 3$, the difference is significant at 5% level of significance.

(*ii*) **Hypothesis.** In the two populations, the proportions of attribute A are not the same, then standard error e of the difference $p_1 - p_2$ is

$$e^2 = e_1^2 + e_2^2 = \frac{p_1\, q_1}{n_1} + \frac{p_2\, q_2}{n_2}, \;\; Z = \frac{p_1 - p_2}{e} < 3,$$

Difference is due to fluctuations of samples.

Example 3. *In a sample of 600 men from a certain city, 450 are found smokers. In another sample of 900 men from another city, 450 are smokers. Do the data indicate that the cities are significantly different with respect to the habit of smoking among men.*

Solution. $n_1 = 600$ men, Number of smokers = 450, $p_1 = \frac{450}{900} = 0.75$

$n_2 = 900$ men, Number of smokers = 450, $p_2 = \frac{450}{900} = 0.5$

$$p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{600 \times 0.75 + 900 \times 0.5}{600 + 900} = \frac{900}{1500} = 0.60$$

$$q = 1 - p = 1 - 0.6 = 0.4$$

$$e^2 = e_1^2 + e_2^2 = pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

$$e^2 = 0.6 \times 0.4\left(\frac{1}{600} + \frac{1}{900}\right) = 0.000667$$

$$e = 0.02582$$

$$Z = \frac{p_1 - p_2}{e} = \frac{0.75 - 0.50}{0.02582} = 9.682$$

$Z > 3$ so that the difference is significant. **Ans.**

Example 4. *One type of aircraft is found to develop engine trouble in 5 flights out of a total of 100 and another type in 7 flights out of a total of 200 flights. Is there a significant difference in the two types of aircrafts so far as engine defects are concerned.*

Solution. $n_1 = 100$ flights, Number of troubled flights = 5, $p_1 = \frac{5}{100} = \frac{1}{20}$

$n_2 = 200$ flights/Number of troubled flights = 7, $p_2 = \frac{7}{200}$

$$e^2 = \frac{p_1\, q_1}{n_1} + \frac{p_2\, q_2}{n_2} = \frac{0.05 \times 0.95}{100} + \frac{0.035 \times 0.965}{200}$$

$$= 0.000475 + 0.0001689 = 0.0006439$$

$$e = 0.0254$$

$$Z = \frac{0.05 - 0.035}{0.0254} = 0.59$$

$Z < 1$, Difference is not significant. **Ans.**

EXERCISE 35.7

1. A random sample of six steel beams has mean compressive strength of 58.392 psi (pounds per square inch) with a standard deviation of s = 648 psi. Test the null hypothesis $H_0 = \mu = 58{,}000$ psi against the alternative hypothesis $H_1 : \mu > 58{,}000$ psi at 5% level of significance (value for t at 5 degree of freedom and 5% significance level is 2.0157). Here m denotes the population mean.

2. A certain cubical die was thrown 96 times and shows 2 upwards 184 times. Is the die biased?

3. In a sample of 100 residents of a colony 60 are found to be wheat eaters and 40 rice eaters. Can we assume that both food articles are equally popular?

4. Out of 400 children, 150 are found to be under weight. Assuming the conditions of simple sampling, estimate the percentage of children who are underweight in, and assign limits within which the percentage probably lies.

5. 500 eggs are taken at random from a large consignment, and 50 are found to be bad. Estimate the percentage of bad eggs in the consignment and assign limits within which the percentage probably lies.

6. A machine puts out 16 imprefect articles in a sample of 500. After the machine is repaired, puts out 3 imprefect articles in a batch of 100. Has the machine been improved?

7. In a city A, 20% of a random sample of 900 school boys had a certain slight physical defect. In another city B > 18.5% of a random sample of 1600 school boys had the same defect. Is the difference between the proportions significant?

8. In two large populations there are 30% and 25% respectively of fair haired people. Is this difference likely to be hidden in samples of 1200 and 900 respectively from the two populations? not hidden at 5% level of significance.

9. One thousand articles from a factory are examined and found to be three percent defective. Fifteen hundred similar articles from a second factory are found to be only 2 percent defective. Can it reasonably be concluded that the product of the first factory is inferior to the second?

10. A manufacturing company claims 90% assurance that the capacitors manufactured by them will show a tolerance of better than 5%. The capacitors are packaged and sold in lots of 10. Show that about 26% of his customers ought to complain that capacitors do not reach the specified standard.

11. An experiment was conducted on nine individuals. The experiment showed that due to smoking, the pulse rate increased in the following order:

$$5, 3, 4, -1, 2, -3, 4, 3, 1.$$

Can you maintain that smoking leads to an increase in the pulse rate?

(t for 8 d.f. at 5% level of significance = 2.31).

12. Nine patients to whom a certain drink was administered registered the following in blood pressure: 7, 3, –1, 4, –3, 5, 6, –4, 1. Show that the data do not indicate that the drink was responsible for these increments.

ANSWERS

2. die is biased. **4.** 37.5% approx. Limits = 37.5 ± 3 (2.4)

5. 10%, 10 ± 3.9 **6.** The machine has not been improved.

7. z = 0.37, Difference between proportions is significant. **8.** z = 2.5

9. It cannot be reasonable concluded that the product of the first factory is inferior to that of the second. **11.** Yes

36

CHAPTER

Tensors Algebra and Applications

36.1 INTRODUCTION

Tensors are logical generalization of vectors. The use of vectors is essential in the mathematical study of a number of physical phenomena. In a similar manner, tensor analysis become popular when Einstein (1879 –1955) used it as a tool for the presentation of his general theory of relativity. It has now become an important mathematical tool in many branches of theoretical physics such as Mechanics, Fluid Mechanics, elasticity, Plasticity, Theory of relativity, electromagnetic theory etc.

The basic principle of tensor calculus is that we should not tie ourselves down to any one system of coordinates. The transformation laws for the components of an entity from one coordinate system to another are the basic criteria to determine the tensor character of that entity.

36.2 SPACE OF N-DIMENSIONS

In a three dimensional rectangular space the coordinates of point are usually denoted by (x, y, z). But this representation of coordinates is not suitable, if we want to generalize the concept of space from rectangular to curvilinear coordinates or from three dimensions to N-dimensions. That is why it is advisable to use a triplet (x^1, x^2, x^3) in place of (x, y, z) where 1, 2, 3 are the **super-scripts** and do not posses any significance as power indices. In general, the coordinates of a point in N-dimensional space, which may or may not be rectangular, are denoted by ***N*-tuples** of the form $(x^1, x^2, .., x^N)$ where 1, 2,..., N are the superscripts for N-variables and not the powers of x. The N-dimensional space is generally denoted by V_N.

Remarks:

(*i*) **A Curve** in the space V_N is defined as the aggregate of points, which satisfy the N-parametric equations.

$$x^i = x^i\,(t),\ (i = 1, 2, \ldots, N) \qquad ...(1)$$

where t is a parameter and $x^i\,(t)$ are N-functions of t, which satisfy certain continuity conditions.

(*ii*) **A subspace** V_M of V_N is defined for $M < N$ as the aggregate of points which satisfy the N-equations

$$x^i = x^i\,(t^1, t^2, \ldots t^M),\ (i = 1, 2, \ldots N). \qquad ...(2)$$

where $t^1, t^2, \ldots t^M$ are M parameters. The $x^i\,(t^1, t^2, \ldots t^M)$ are N-functions of $t^1, t^2, \ldots t^M$ which satisfy certain continuity conditions. When $M = N - 1$, the subspace is called **hypersurface.**

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36.3 COORDINATES TRANSFORMATION

Let. $(x^1, x^2, .., x^N)$ and $(\bar{x}^1, \bar{x}^2, .., \bar{x}^N)$ be coordinates of a point in two different frames of reference in a V_N. Suppose there exist *N*-independent relations between the coordinates of the two systems having the form

$$\bar{x}^j = \bar{x}^j (x^1, x^2, .., x^N), \ (j = 1, 2, ..., N) \qquad ...(3)$$

where it is assumed that the functions involved are single valued continuous and have continuous derivatives. Then the above set of *N*-equations may be solved and to each set of coordinates $(\bar{x}^1, \bar{x}^2, .., \bar{x}^N)$ there will correspond a unique set $(x^1, x^2, .., x^N)$ given by

$$x^i = x^i (\bar{x}^1, \bar{x}^2, .., \bar{x}^N), \ i = 1, 2, ..., N. \qquad ...(4)$$

The relations (3) and (4) define a transformation of coordinates from one frame of reference to another. Differentiating (3) wet get

$$d\bar{x}^j = \frac{\partial \bar{x}^j}{\partial x^1} dx^1 + \frac{\partial \bar{x}^j}{\partial x^2} dx^2 + ... + \frac{\partial \bar{x}^j}{\partial x^N} dx^N$$

$$= \sum_{i=1}^{N} \frac{\partial \bar{x}^j}{\partial x^i} dx^i \qquad ...(5)$$

This is the *coordinate differential transformation rule, i.e., the change in the direction of coordinates*.

Note : Throughout the text, to denote the coordinates of a paint only a superscript will be used.

To write the results in compact form, which is the prime aim of tensor analysis, we introduce the following two conventions :

(*i*) **Indicial convention :**

In a *N*-dimensional space, indices used either as subscripts or superscripts will take all values from 1 to *N*-unless otherwise stated.

Hence, the equations (3) and (4) may be written as

$$\bar{x}^j = \bar{x}^j (x^i), \qquad x^i = x^i (\bar{x}^j), \text{ respectively.} \qquad ...(6)$$

(*ii*) **Einstein's summation convention :**

In a N-dimensional space, if an index is repeated in a term then it implies summation with respect to that index over the range 1, 2, ..., *N unless the contrary is specified.*

Hence, using summation convention, the relation (5) for the coordinate differential transformation may be written as

$$d\bar{x}^j = \frac{\partial \bar{x}^j}{\partial x^i} dx^i \qquad ...(7)$$

Similarly, by differentiating (4), wet get

$$dx^i = \frac{\partial x^i}{\partial \bar{x}^j} d\bar{x}^j \qquad ...(8)$$

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Remarks:

(1) *The repeated index is called a* **dummy index**, *as it can be replaced by any other index not used in that term. As for example, equation* (7) *may equally well be written as*

$$d\overline{x}^j = \frac{\partial \overline{x}^j}{\partial x^k} dx^k$$

or

$$d\overline{x}^j = \frac{\partial \overline{x}^j}{\partial x^l} dx^l$$

Similarly, we may write equation (8) as

$$dx^i = \frac{\partial x^i}{\partial \overline{x}^m} d\overline{x}^m$$

Or any other superscript in place of m. This device of changing in dummies is often employed as a useful manipulative trick for simplifying expressions. But the index j in equation (7) and index i in equation (8) are not repeated and are called free **indices**.

(2) It may be noted (rather remembered) that the free indices on the two sides of an equation must be the same.

(3) We shall use brackets, usually, to indicate powers. Thus the square of x^N will be written as $(x^N)^2$.

(4) To avoid confusion the same index must not be used more than twice in any single term. For example $\left(\sum_{i=1}^{N} A_i x^i\right)^2$ will not be written as $A_i x^i A^i x^i$ but rather $A_i A_j x^i x^j$. The difference in the use of superscripts and subscripts will be explained in due course.

36.4 KRONECKER DELTA

The Kronecker delta is written as δ^i_j and is defined by

$$\delta^i_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \qquad ...(9)$$

Thus, $\delta^1_1 = \delta^2_2 = ... = \delta^N_N = 1$ (no summation over N)

$$\delta^1_2 = \delta^2_3 = ... = 0,$$

and,

$$\delta^i_i = \delta^1_1 + \delta^2_2 + ... + \delta^N_N$$

$$= 1 + 1 + ... + 1 = N \qquad ...(10)$$

An important property of Kronecker delta is that

$$\delta^i_j A^j = A^i, \qquad ...(11)$$

since in the left-hand side of this equation when summation is carried out over; the only surviving term will be one for which $j = i$. *This shows that the role of* δ^i_j *when it is multiplied with an entity, is to replace the index j in the entity by i or vice versa and then itself falls out.*

It may be noted that

$$\frac{\partial x^i}{\partial x^j} = \delta^i_j \qquad ...(12)$$

because the coordinates x^i are independent. Similarly,

$$\frac{\partial \bar{x}^i}{\partial \bar{x}^j} = \delta^i_j \qquad ...(13)$$

Example 1. *Write each of the following, using Einstein's summation convention*

(*i*) $A_1^k B^1 + A_2^k B^2 + ... + A_N^k B^N$

(*ii*) $ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + ...+ g_{NN}(dx^N)^2 + g_{12}dx^1dx^2$
$+ g_{21}dx^2dx^1 + ... + g_{1N}\, dx^1dx^N + g_{N1}\, dx^N dx^1$

Solution:

(*i*) $A_1^k B^1 + A_2^k B^2 + ... + A_N^k B^N = A_i^k B^i$ (*ii*) $ds^2 = g_{ij}dx^i dx^j$ **Ans.**

Example 2. *Show that*

(*i*) $\delta^i_j\ \delta^j_k = \delta^i_k$ (*ii*) $\dfrac{\partial x^k}{\partial \bar{x}^i}\dfrac{\partial \bar{x}^i}{\partial x^j} = \delta^k_j$

Solution:

(*i*) $\delta^i_j\ \delta^j_k = \delta^i_1\ \delta^1_k + \delta^i_2\ \delta^2_k + ... + \delta^i_k\ \delta^k_k + ... + \delta^i_N\ \delta^N_k$

$= 0 + 0 + ... + \delta^i_k\,(1) + ... + 0$ (no summation over k)

$= \delta^i_k$.

(*ii*) $\dfrac{\partial x^k}{\partial \bar{x}^i}\dfrac{\partial \bar{x}^i}{\partial x^j} = \dfrac{\partial x^k}{\partial \bar{x}^1}\dfrac{\partial \bar{x}^1}{\partial x^j} + \dfrac{\partial x^k}{\partial \bar{x}^2}\dfrac{\partial \bar{x}^2}{\partial x^j} + ... + \dfrac{\partial x^k}{\partial \bar{x}^N}\dfrac{\partial \bar{x}^N}{\partial x^j}$

$= \dfrac{\partial x^k}{\partial x^j}$ (by chain rule of partial differentiation)

$= \delta^k_j$. [using (12)] ...(14)

Example 3. *If* $a_{\alpha\beta}x^\alpha x^\beta \equiv 0$ *for all values of the variables* $x^1, x^2,.., x^N$; *show that* $a_{ij} + a_{ji} = 0$.

Solution: Let $S = a_{\alpha\beta}x^\alpha x^\beta \equiv 0$

then $\dfrac{\partial S}{\partial x^i} = a_{\alpha\beta}x^\alpha \delta^\beta_i + a_{\alpha\beta}\delta^\alpha_i x^\beta = 0$

$= a_{\alpha i}\, x^\alpha + a_{i\beta}\, x^\beta = 0$

Now, differentiation with respect to x^j gives

$$\frac{\partial^2 S}{\partial x^j \partial x^i} = a_{\alpha i}\ \delta^\alpha_i + a_{i\beta}\ \delta^\beta_j = 0$$

$$= a_{ij} + a_{ji} = 0.$$

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36.5 SUMMATION OF CO-ORDINATES

The equations of co-ordinates can be written in very compact form in terms of summation convention. We write (x_1, x_2, x_3) and $(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ instead of (x, y, z) and (x', y', z') and denote the co-ordinate axes as OX_1, OX_2, OX_3 and $O\overline{X}_1, O\overline{X}_2, O\overline{X}_3$. Also we denote x_i, $\overline{x}_j$ as the co-ordinates of a point P relative to the two systems of axes; where $i = 1, 2, 3, j = 1, 2, 3$. Let l_{ij} denote the cosines of the angles between $OX_i, O\overline{X}_j$. In general $l_{ij} \neq l_{ji}$

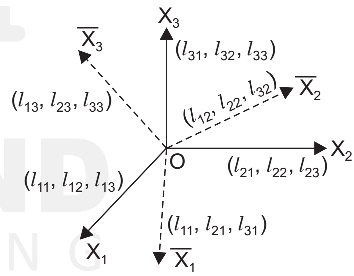
The equation of co-ordinate transformation can be written as

$$\left.\begin{aligned} \overline{x}_1 &= l_{11}x_1 + l_{21}x_2 + l_{31}x_3 \\ \overline{x}_2 &= l_{12}x_1 + l_{22}x_2 + l_{32}x_3 \\ \overline{x}_3 &= l_{13}x_1 + l_{23}x_2 + l_{33}x_3 \end{aligned}\right\} \quad ...(1a)$$

$$\left.\begin{aligned} x_1 &= l_{11}\overline{x}_1 + l_{12}\overline{x}_2 + l_{13}\overline{x}_3 \\ x_2 &= l_{21}\overline{x}_1 + l_{22}\overline{x}_2 + l_{23}\overline{x}_3 \\ \overline{x}_3 &= l_{31}\overline{x}_1 + l_{32}\overline{x}_2 + l_{33}\overline{x}_3 \end{aligned}\right\} \quad ...(1b)$$

These equations of co-ordinate transformation can be represented by means of a table form such that

	x_1	x_2	x_3
$\overline{x}_1$	l_{11}	l_{21}	l_{31}
$\overline{x}_2$	l_{12}	l_{22}	l_{32}
$\overline{x}_3$	l_{13}	l_{23}	l_{33}



Adopting summation on convention *i.e*,

$$a_{11} + a_{22} + a_{33} = a_{ij}$$

$a_{ip}\, b_{iq} = a_{1p}\, b_{1q} + a_{2p}\, b_{2q} + a_{3p}\, b_{3q}$ we re-write above equations as

$$\overline{x}_1 = l_{i1}\, x_i \qquad x_1 = l_{1j}\, \overline{x}_j$$

$$\overline{x}_2 = l_{i2}\, x_i \qquad x_2 = l_{2j}\, \overline{x}_j$$

$$\overline{x}_3 = l_{i3}\, x_i \qquad x_3 = l_{3j}\, \overline{x}_j$$

We can re-write these equations in single equation in the form.

$$\overline{x}_j = l_{ij}\, x_i,\ x_i = l_{ij}\, \overline{x}_j$$

which are complete equivalents of the equations of co-ordinate transformation from either system to another.

36.6 RELATION BETWEEN THE DIRECTION COSINES

The direction cosines of any three mutually perpendicular straight lines $O\overline{X}_1, O\overline{X}_2, O\overline{X}_3$ relative to the system OX_1, OX_2, OX_3 are l_{11}, l_{21}, l_{31}, l_{12}, l_{22}, l_{32}, l_{23}, l_{33}.

The relation between these direction cosines are

$l_{11}\, l_{11} + l_{21}\, l_{21} + l_{31}\, l_{31} = l_{j1}\, l_{j1} = l_j\, l_{12}\, l_{12} + l_{22}\, l_{22} + l_{32}\, l_{32} = l_{j2}\, l_{j2} = 1$

$l_{13}\, l_{13} + l_{23}\, l_{23} + l_{33}\, l_{33} = l_{j3}\, l_{j3} =$

Similarly,

$l_{11}\, l_{12} + l_{21}\, l_{22} + l_{31}\, l_{32} = l_{j1}\, l_{j2} = 0,\; l_{12}\, l_{13} + l_{22}\, l_{23} + l_{32}\, l_{33} = l_{j2}\, l_{j3} = 0$

$l_{13}\, l_{11} + l_{23}\, l_{21} + l_{33}\, l_{31} = l_{j3}\, l_{j1} = 0$

Finally, we can write these equations by means of a single equation as

$$l_{ij} l_{kj} = \begin{cases} 1, & \text{when } i = k \\ 0, & \text{when } i \neq k \end{cases} \qquad \text{or} \qquad \delta_{ik} = \begin{cases} 1, & \text{when } i = k \\ 0, & \text{when } i \neq k \end{cases}$$

where δ_{ik} is the kronecker delta.

or $$\delta_{ik} = l_{ij}\, l_{kj}$$

Now, we know that $\bar{x}_j = l_{ij}\, x_i$

Multiplying both sides by l_{jk} then

or $$l_{jk}\, \bar{x}_j = l_{ij}\, l_{kj}\, x_x \quad \Rightarrow \quad l_{jk}\, \bar{x}_j = \delta_{ik}\, x_i$$

putting $i = k$ *i.e.*, $\delta_{ik} = 1$ when $i = k$

$$\delta_{kk}\, x_k = l_{jk}\, \bar{x}_j \;\Rightarrow\; x_k = l_{jk}\, \bar{x}_j$$

DEFINITION AND TYPES OF TENSORS

36.7 CONTRAVARIANT VECTORS (CONTRAVARIANT TENSORS OF FIRST ORDER)

Motivated by the relation (7), we give the following definition of contravariant vectors :

Definition : *If a set of N quantities A^i in a coordinate system x^i are related to another N quantities $\bar{A}^j$ in a coordinate system $\bar{x}^j$ by the trans formation equations*

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^q} A^q, \text{ (Contravariant Law)} \qquad ...(15)$$

then A^i (read as A superscript i) are said to be the components of a contravariant vector or contravariant tensor of the first order (or first rank).

Notes : (*i*) *The components of contravariant tensors are denoted by superscripts as a convention, with the exception of the coordinates x^i, which may behave as a contravariant vector in special conditions.*

(*ii*) *The tensor entity itself may be denoted by a bold faced letter* **A**, *but it is advisable to use its components such as A^i rather than* **A** *which, as we shall see later, will help us in knowing its order by the superscripts or subscripts.*

Multiplying equation (15) by $\dfrac{\partial x^r}{\partial \bar{x}^p}$ and summing over the index p from 1 to N, we find

$$\frac{\partial x^r}{\partial \bar{x}^p} \bar{A}^p = \frac{\partial x^r}{\partial \bar{x}^p} \frac{\partial \bar{x}^p}{\partial x^q} A^q$$

$$= \frac{\partial x^r}{\partial \bar{x}^p} A^q = \delta^r_q A^q = A^r$$

or $$A^r = \frac{\partial x^r}{\partial \bar{x}^p} \bar{A}^p \qquad \textbf{(Contravariant Law)} \qquad ...(16)$$

Thus equation (16) may equally well be taken as the transformation law for contravariant vectors.

From equation (7), in view of transformation law, we conclude that the coordinate *differential* dx^i *form the components a contravariant vector*. It follows immediately that $\frac{d\,x^i}{d\,t}$ is also a contravariant vector, called the tangent vector to the curve $x^i = x^i(t)$.

In general an entity whose components transform as the *coordinate differential transformation rule* (**unlike** *i.e.*, in contrast to the partial differentiation transformation rule of a scalar function) is called an entity having *contravariant components* or in short contravariant entity.

Theorem *The law of transformation for a contravariant vector is transitive.*

Proof: Let the components of a contravariant vector relative to the coordinate system x^i be A^i and relative to the coordinate system $\bar{x}^j$ be $\bar{A}^j$. Then by contravariant law of transformation

$$\bar{A}^j = \frac{\partial \bar{x}^j}{\partial x^i} A^i \qquad ...(17)$$

Now, a further change of coordinates from $\bar{x}^j$ to x'^k, the. new components A'^k by contravariant law must be given by

$$A'^k = \frac{\partial x'^k}{\partial \bar{x}^j} \bar{A}^j \qquad ...(18)$$

Combining (17) and (18), we get

$$A'^k = \frac{\partial x'^k}{\partial \bar{x}^j} \frac{\partial \bar{x}^j}{\partial x^i} A^i$$

$$= \frac{\partial x'^k}{\partial x^i} A^i \qquad ...(19)$$

This shows that the transformation law of contravariant vector is transitive.

Theorem 2. *The coordinates* x^i *behave like the components of a contravariant vector with respect to linear transformation of the type* $\bar{x}^j = a_i^j x^i$, where a_i^j *are a set of* N^2 *constants*.

Proof : Since, $$\bar{x}^j = a_i^j x^i, \qquad ...(20)$$

On differentiation, we get

$$\frac{\partial \bar{x}^j}{\partial x^i} = a_i^j \qquad ...(21)$$

a_i^j being constants.

Combining (20) and (21) we find

$$\bar{x}^j = \frac{\partial \bar{x}^j}{\partial x^i} x^i \qquad ...(22)$$

Hence the proposition,

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Example 4. (*i*) *If a vector has components* $\dot{x}, \dot{y}$ $\left(\dot{x} = \frac{dx}{dt}, \dot{y} = \frac{dy}{dt}\right)$ *in rectangular cartesian coordinates then* $\dot{r}, \dot{\theta}$ *are its components in polar coordinates.*

(*ii*) *A vector has components* $\ddot{x}, \ddot{y}$ *in rectangular cartesian coordinates then its respective components in polar coordinates are*

$$\ddot{r} - r\dot{\theta}^2, \ddot{\theta} + \frac{2}{r} r\dot{\theta}.$$

Solution: Here, the space is two-dimensional,

$$x^1 = x, \qquad x^2 = y$$

$$\bar{x}^1 = r, \qquad \bar{x}^2 = \theta \qquad \text{...(23)}$$

$$r^2 = x^2 + y^2 \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

(*i*) Since, $\dot{x} = \frac{dx}{dt} = \frac{dx^1}{dt}$

and $\dot{y} = \frac{dy}{dt} = \frac{dx^2}{dt}$ are contravariant vectors,

we take $A^1 = \dot{x}, \quad A^2 = \dot{y}$ ...(24)

and use **Contravariant law**, *viz*.,

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j = \frac{\partial \bar{x}^i}{\partial x^1} A^1 + \frac{\partial \bar{x}^i}{\partial x^2} A^2$$

to get,

$$\bar{A}^1 = \frac{\partial \bar{x}^1}{\partial x^1} A^1 + \frac{\partial \bar{x}^1}{\partial x^2} A^2$$

$$= \frac{\partial r}{\partial x}\dot{x} + \frac{\partial r}{\partial y}\dot{y}$$

$$= \frac{x}{r}\dot{x} + \frac{\partial r}{\partial y}\dot{y}$$

$$= \frac{x\dot{x} + y\dot{y}}{r} = \frac{r\dot{r}}{r} = \dot{r} \qquad \text{...(25)}$$

and,

$$\bar{A}^1 = \frac{\partial \bar{x}^2}{\partial x^1} A^1 + \frac{\partial \bar{x}^2}{\partial x^2} A^2$$

$$= \frac{\partial r}{\partial x}\dot{x} + \frac{\partial r}{\partial y}\dot{y}$$

$$= -\frac{y}{r^2}\dot{x} + \frac{x}{r^2}\dot{y}$$

$$= \frac{x\dot{y} + y\dot{x}}{r^2} = \dot{\theta} \qquad \text{...(26)}$$

(*ii*) Similarly taking,

$$A^1 = \ddot{x}, \quad A^2 = \ddot{y} \qquad \text{...(27)}$$

we find, $\bar{A}^1 = \frac{\partial r}{\partial x}\ddot{x} + \frac{\partial r}{\partial y}\ddot{y}$

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$$= \frac{x\,\ddot{x} + y\,\ddot{y}}{r} = \frac{r\,\ddot{r} - r^2\dot{\theta}^2}{r}$$

$$= \ddot{r} - r\dot{\theta}^{2*} \qquad \text{...(28)}$$

and
$$\overline{A}^2 = \frac{\partial\theta}{\partial x}\ddot{x} + \frac{\partial\theta}{\partial y}\ddot{y}$$

$$= \frac{x\,\ddot{y} - y\,\ddot{x}}{r^2}$$

$$= \frac{r^2\ddot{\theta} + 2r\dot{r}\,\dot{\theta}}{r^2} \quad \text{(Differentiating III)*}$$

$$= \ddot{\theta} + \frac{2}{r}\dot{r}\,\dot{\theta} \qquad \text{...(29)}$$

It may he noted that the velocity and acceleration components are Contravariant Vectors.

* since	$x^2 + y^2 = r^2$, we have $x\dot{x} + y\dot{y} = r\dot{r}$	...(I)
	$x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 = r\ddot{r} + \dot{r}^2$	...(II)
Also	$\theta = \tan^{-1}\dfrac{y}{x}$, therefore $r^2\dot{\theta} = x\dot{y} - y\dot{x}$	...(III)
from (I) and (III)	$r^2\dot{r}^2 + r^4\dot{\theta}^2 = (x^2 + y^2)(\dot{x}^2 + \dot{y}^2)$	
or	$\dot{r}^2 + r^2\dot{\theta}^2 = \dot{x}^2 + \dot{y}^2$	...(IV)
final result follows from (II) and (IV)		

Further, the difference between resolved parts and components may also be observed because $\dot{r}, \dot{\theta}$ are components of the velocity vector but its resolved, parts are $\dot{r}$ and $r\dot{\theta}$, as they have the same dimensions. In cartesian system, the resolved parts and com-ponents are indistinguishable while in oblique system they may differ.

36.8 INVARIANT

A function I of the N coordinates x^i [$I = I(x^i)$] is called an invariant or a scalar function or tensor of order zero with respect to coordinate transformations if

$$I = \overline{I} \qquad \text{...(30)}$$

where $\overline{I}$ [$\overline{I} = \overline{I}$ ($\overline{x}^j$)] is the value of I in the new coordinate system $\overline{x}^j$.

Note : If ϕ is an invariant function of the coordinate x^i *i.e.*, $\phi = \phi(x^i)$, then on transformation of coordinates to $\overline{x}^j$, we have

$$\frac{\partial\phi}{\partial\overline{x}^j} = \frac{\partial\phi}{\partial x^i}\frac{\partial x^i}{\partial\overline{x}^j}$$

or
$$\left(\frac{\partial\phi}{\partial\overline{x}^j}\right) = \frac{\partial x^i}{\partial\overline{x}^j}\left(\frac{\partial\phi}{\partial x^i}\right) \qquad \text{...(31)}$$

This law of transformation is rather like (15), but the partial derivative involving the two sets of coordinates is the other way up. This shows that we may have another type of quantities which transform in the manner of (31).

This is the transformation rule of the *partial derivative of an invariant* which is a function of the-coordinate x^i (change in the direction orthogonal to constant coordinate surfaces).

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36.9 COVARIANT VECTORS (COVARIANT TENSORS OF FIRST ORDER)

Motivated by the relation (31), we give the following definition of Covariant Vectors :

Definition : *If a set of N quantities A_i in a coordinate system x^i are related to another N quantities* $\overline{A}^j$ in a *coordinate system* $\overline{x}^j$ *the transformation equations*

$$\overline{A}_p = \frac{\partial x^q}{\partial \overline{x}^p} A_q \text{, } (\textbf{Covariant law}) \qquad ...(32)$$

then A_i (read as A subscript i) are said to be the components of a Covariant Vector or covariant tensor of the first order or first rank.

Multiplying equation (32) by $\frac{\partial \overline{x}^p}{\partial x^r}$ and summing over the index p from 1 to N, we find

$$\frac{\partial \overline{x}^p}{\partial x^r} \overline{A}_p = \frac{\partial \overline{x}^p}{\partial x^r} \frac{\partial x^q}{\partial \overline{x}^p} A_q$$

$$= \frac{\partial x^q}{\partial x^r} A_q = \delta^q_r A_q = A_r$$

or

$$A_r = \frac{\partial \overline{x}^p}{\partial x^r} \overline{A}_p \quad (\textbf{Covariant law}) \qquad ...(33)$$

Thus relation (33) may equally well be taken as the transformation law of Covariant Vectors.

Notes :

(*i*) *The components of covariant tensors are denoted by subscripts as a convention.*

(*ii*) *It follows immediately that the quantities* $\frac{\partial \phi}{\partial x^i}$ *in equation* (31) *are the components of a Covariant Vector, where in conformity with the convention the index i in* $\frac{\partial \phi}{\partial x^i}$ *is regarded as a subscript. Such a Covariant Vector is called the gradient of* ϕ.

(*iii*) *In general, an entity whose components transform like i.e., in conformity to the transformation rule of the partial derivative of an invariant of coordinate function, is called an entity having covariant components or in short covariant entity.*

(*iv*) *The contravariance and covariance of an entity is not the intrinsic* (*inherent*) *property of the entity but this distinction is due to the way in which the components of the entity are related to the coordinate system to which it is referred. For example, when the components of a velocity vector are taken along the coordinate axes* (*Example* 4) *they are contravariant components and if it is represented as the gradient of a potential function* (*scalar function*), *which are perpendicular to the coordinate axes, then these components are covariant.*

In fact we can transform from one set of components of a given vector to another by means of the metric tensor,

Theorem 3. ***There exists no distinction between Contravariant and Covariant Vectors when we restrict ourselves to coordinate transformations of the type***

$$\overline{x}^i = a^i_m \, x^m + b^i \text{,}$$

where b^i are N constants which do not necessarily form the components of a Contravariant Vector and a_m^i are N^2 constants which do not necessarily form the components of a tensor such that

$$a_r^i\, a_m^i = \delta_m^r\,.$$

We have

$$\bar{x}^i = a_m^i\, x^m + b^i \qquad ...(34)$$

Multiplying by a_r^i, and summing over the index i we get

$$a_r^i\, \bar{x}^i = a_r^i\, a_m^i\, x^m + a_r^i\, b^i$$

and using the given relation, *viz*.

$$a_r^i\, a_m^i = \delta_m^r \qquad ...(35)$$

we find

$$a_r^i \bar{x}^i = \delta_m^r x^m + a_r^i b^i$$

$$= x^r + a_r^i\, b^i.$$

Finally, replacing the free index r by m on both sides, we obtain

$$x^m = a_m^i \bar{x}^i - a_m^i b^i. \qquad ...(36)$$

From (34) and (36) it follows that

$$\frac{\partial \bar{x}^i}{\partial x^m} = a_m^i = \frac{\partial x^m}{\partial \bar{x}^i}\cdot \qquad ...(37)$$

This shows that (15) and (32), transformation laws for Contravariant and Covariant Vectors respectively, define the same type of entity in the present case. This is in fact the rectangular cartesian transformation of coordinates (*orthogonal Euclidean space*).

Theorem 4. The law of transformation for a Covariant Vector is transitive.

Proof : Let the components of a Covariant Vector relative to the coordinate system x^i be A_i and relative to the coordinate system $\bar{x}^j$ be $\bar{A}^j$. Then by covariant law of transformation.

$$\bar{A}_j = \frac{\partial x^i}{\partial \bar{x}^j} A_i\,. \qquad ...(38)$$

Now for a further change of coordinates from $\bar{x}^j$ to x'^k, the new components A'_k by covariant law must be given by

$$A'_k = \frac{\partial \bar{x}^j}{\partial x'_k} \bar{A}^j \qquad ...(39)$$

Combining (38) and (39) we get

$$A'_k = \frac{\partial \bar{x}^j}{\partial x'_k} \frac{\partial x^i}{\partial \bar{x}^j} A_i = \frac{\partial x^i}{\partial x'_k} A_i \qquad ...(40)$$

Hence the proposition.

Example 5. *A Covariant tensor of first order has components xy, $2y - z^2$, xz in rectangular coordinates. Determine its Covariant components in spherical polar coordinates.*

Solution: Here the space is three dimensional,

$$x^1 = x,\ x^2 = y,\ x^3 = z, \qquad \overline{x}^1 = r, \overline{x}^2 = \theta, \overline{x}^3 = \phi\ .$$

Further,

$$x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r\cos\theta \qquad ...(41)$$

Taking, for the given covariant vector A_i,

$$A_1 = xy,\ A_2 = 2y - z^2,\ A_3 = xz \qquad ...(42)$$

and using Covariant law, *viz*.

$$\overline{A}_i = \frac{\partial x^j}{\partial \overline{x}^i} A_j$$

$$= \frac{\partial x^1}{\partial \overline{x}^i} A_1 + \frac{\partial x^2}{\partial \overline{x}^i} A_2 + \frac{\partial x^3}{\partial \overline{x}^i} A_3 \qquad ...(43)$$

We find
$$\overline{A}_1 = \frac{\partial x^1}{\partial \overline{x}^i} A_1 + \frac{\partial x^2}{\partial \overline{x}^i} A_2 + \frac{\partial x^3}{\partial \overline{x}^i} A_3$$

$$= \frac{\partial x}{\partial r}(xy) + \frac{\partial y}{\partial r}\left(2y - z^2\right) + \frac{\partial z}{\partial r}(xz) \qquad ...(44)$$

using (41), we finally obtain

$$\overline{A}_1 = \left(\sin\theta\cos\phi\right) r^2 \sin^2\theta \sin\phi\cos\phi$$
$$+ (\sin\theta \sin\phi)\ (2r \sin\theta \sin\phi - r^2 \cos^2\theta)$$
$$+ (\cos\theta)\ r^2 \sin\theta \cos\theta \cos\phi \qquad ...(45)$$

Similarly, from (43)

$$\overline{A}_2 = \frac{\partial x^1}{\partial \overline{x}^2} A_1 + \frac{\partial x^2}{\partial \overline{x}^2} A_2 + \frac{\partial x^3}{\partial \overline{x}^3} A_3$$

$$= \frac{\partial x}{\partial \theta}(xy) + \frac{\partial y}{\partial \theta}\left(2y - z^2\right) + \frac{\partial z}{\partial \theta}(xz)$$

$$= (r\cos\theta\cos\phi)\ r^2 \sin^2\theta \sin\phi \cos\phi$$
$$+ (r\cos\theta \sin\phi)\ (2r\sin\theta\sin\phi - r^2\cos^2\theta)$$
$$- (r\sin\theta)\ (r^2 \sin\theta\cos\theta\cos\phi) \qquad ...(46)$$

and, $\overline{A}_3 = \left(-r\sin\theta\sin\phi\right) r^2 \sin^2 \sin\phi\cos\phi$

$$+ (r\sin\theta\cos\phi)\ (2r\sin\theta\sin\phi - r^2\cos^2\theta) \qquad ...(47)$$

Product of Vectors

Product of two Vectors*

(*a*) Product of two contravariant Vectors :

Let the components of two Contravariant vectors relative to the coordinate system x^i be A_i and B^j and relative to the coordinate system $\overline{x}^m$ the components $\overline{A}^p$ and $\overline{B}^q$. Then by contravariant law

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$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^i} A^i \qquad ...(48)$$

and
$$\bar{B}^q = \frac{\partial \bar{x}^q}{\partial x^j} B^j \qquad ...(49)$$

Multiplying (48) and (49), known as outer product, we get

$$\bar{A}^p \ \bar{B}^q = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} A^i B^j \qquad ...(50)$$

If we denote the N^2 quantities $A^i B^j$ by C^{ij} and $\bar{A}^p \ \bar{B}^q$ by $\bar{C}^{pq}$ then

$$\bar{C}^{pq} = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} C^{ij} \qquad ...(51)$$

(*b*) Product of two Covariant Vectors :

Let the components of two Covariant vectors relative to the coordinate system x^i be A_i and B^j and relative to $\bar{x}^m$ be $\bar{A}_p$ and $\bar{B}_q$.

Then by Covariant law

$$\bar{A}_p = \frac{\partial x^i}{\partial \bar{x}^p} A_i \qquad ...(52)$$

and
$$\bar{B}_p = \frac{\partial x^j}{\partial \bar{x}^q} B_j \qquad ...(53)$$

Hence
$$\bar{A}_p \ \bar{B}_q = \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q} A_i B_j$$

or
$$\bar{C}_{pq} = \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q} C_{ij} \qquad ...(54)$$

(*c*) Product of a Contravariant Vector and a Covariant Vector:

Let the components of Contravariant vector and a Covariant vector relative to the coordinate system x^i be A_i and B^j respectively and relative to $\bar{x}^m$ be $\bar{A}^p$ and $\bar{B}_q$ respectively. Then

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^i} A^i \qquad ...(55)$$

and
$$\bar{B}_p = \frac{\partial x^j}{\partial \bar{x}^q} B_j \qquad ...(56)$$

Hence
$$\bar{A}^p \ \bar{B}_q = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^q} A^i B_j \qquad ...(57)$$

Denoting $A^i \ B_j$ by C^i_j and $\bar{A}^p \ \bar{B}_q$ by $\bar{C}^p_q$, we find

$$\bar{C}^p_q = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^q} C^i_j \qquad ...(58)$$

Note : *Relations* (51), (54) *and* (58) *show that the* N^2 *components of each* C^{ij}, C_{ij} and C^i_j *satisfy different types of transformation laws when transformed from one coordinate system to another and encourage us to define new tensor entities.*

36.10 SECOND ORDER TENSORS

(*a*) Contravariant tensor of the second order :

Motivated by the relation (51) we give the following definition of second order contravariant tensor.

Definition : *If a set of* N^2 *quantities* A^{ij} *in a coordinate system* x^i *are related to another* N^2 *quantities* $\overline{A}^{kl}$ *relative to the coordinate system* $\overline{x}^j$ *by the transformation equations*

$$\overline{A}^{kl} = \frac{\partial \overline{x}^k}{\partial x^i}\frac{\partial \overline{x}^l}{\partial x^j} A^{ij}, \quad \textbf{(Contravariant law)} \qquad ...(59)$$

then A^{ij} *are said* to be the components of a **contravariant tensor of the second order** (*or second rank*).

(*b*) Covariant tensor of the Second order :

Motivated by the relation (54), we give the following definition of second order covariant tensor :

Definition : *If a set of* N^2 *quantities* A_{ij} *in a coordinate system* x^i *are related to another* N^2 *quantities* $\overline{A}_{kl}$ *relative to the coordinate system* $\overline{x}^j$ *by the transformation equations*

$$\overline{A}_{kl} = \frac{\partial x^i}{\partial \overline{x}^k}\frac{\partial x^j}{\partial \overline{x}^l} A_{ij} \quad \textbf{(Covariant law)} \qquad ...(60)$$

then A_{ij} *are said to be the components of a* **Covariant tensor of the second order** (*or second rank*).

(*c*) **Mixed tensor of the second order :**

Motivated by the relation (58), we give the following definition of a second order mixed tensor :

Definition : *If a set of* N^2 *quantities* A^i_j *in a coordinate system* x^i *are related to another* N^2 *quantities* $\overline{A}^i_j$ *relative to the coordinate system* $\overline{x}^j$ *by the transformation equations*

$$\overline{A}^k_l = \frac{\partial \overline{x}^k}{\partial x^i}\frac{\partial x^j}{\partial \overline{x}^l} A^i_j \quad \textbf{(mixed tensor law)} \qquad ...(61)$$

then A^i_j *are said to be the components of a* **mixed tensor of the second order** (*or second rank*).

Remarks:

(*i*) *It may be noted that the indices are placed on the tensors as* **superscripts** *to denote* **contravariance** *and as* **subscripts** *to denote* **covariance**. *Thus a mixed tensor* A^i_j *is contravariant in i and covariant in j and transform accordingly.*

(*ii*) *It is now obvious that* A^i *and* B^j *are the components of two contravariant tensors of first order and* A_i *and* B_j *are the components of two covariant tensors of first order then*

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(α) A^iB^j is a **Contravariant tensor of the second order**

(β) A_iB_j is a **Covariant tensor of the second order**

(γ) A^iB_j and A_iB^j are **Mixed tenors of the second order**

Theorem 5. ***The kronecker delta is a mixed tensor of the second order whose components in any other coordinate' system again constitute the kronecker delta.***

Poof : The kronecker delta is

$$\delta^i_j = \begin{cases} 1, & if\ i = j \\ 0, & if\ i \neq j \end{cases} \qquad ...(62)$$

Let δ^i_j be the components in the coordinate system x^i and the corresponding components in $\overline{x}^i$ be δ^i_j. If we can prove that these components obey the transformation law (61) of mixed tensors, then it will be a mixed tensor.

Now,
$$\frac{\partial \overline{x}^k}{\partial x^i}\frac{\partial x^j}{\partial \overline{x}^l}\delta^i_j = \frac{\partial \overline{x}^k}{\partial x^i}\frac{\partial x^j}{\partial \overline{x}^l} = \frac{\partial \overline{x}^k}{\partial \overline{x}^l} = \delta^k_l \qquad ...(63)$$

Hence the proposition.

Notes :

(*i*) *If kronecker delta is defined as*,

$$\delta_{ij} = \begin{cases} 1, & if\ i = j \\ 0, & if\ i \neq j \end{cases}$$

then it is not a covariant tensor, since the transformed components $\frac{\partial x^i}{\partial \overline{x}^k}\frac{\partial x^j}{\partial \overline{x}^l}\delta_{ij} = \frac{\partial x^j}{\partial \overline{x}^k}\frac{\partial x^j}{\partial \overline{x}^l}$ does not yield kronecker delta.

(*ii*) *A tensor which has the same set of components relatively to every system of coordinate axes is called an* **isotropic tensor**.

Clearly kronecker tensor is an isotropic tensor.

$$\left(\overline{\delta}^i_j = \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial x^q}{\partial \overline{x}^j}\delta^p_q = \frac{\partial \overline{x}^i}{\partial x^q}\frac{\partial x^q}{\partial \overline{x}^j} = \frac{\partial \overline{x}^i}{\partial \overline{x}^j} = \delta^i_j\right)$$

Hence, $\overline{\delta}^i_j = \delta^i_j$, *i.e.*, it has constant components independent of coordinate axes.

Example 6. *If A_{ij} is a covariant tensor of the second order and B^i, C^j are contravariant vectors; prove that $A_{ij}B^iC^j$ is, an invariant.*

Solution: We have

$$\overline{A}_{kl} = \frac{\partial x^i}{\partial \overline{x}^k}\frac{\partial x^j}{\partial \overline{x}^l}A_{ij} \qquad ...(64)$$

$$\overline{B}^k = \frac{\partial \overline{x}^k}{\partial x^p}B^p \qquad ...(65)$$

$$\overline{C}^l = \frac{\partial \overline{x}^l}{\partial x^q}C^q \qquad ...(66)$$

then $$\bar{A}_{kl}\ \bar{B}^k\ \bar{C}^l = \frac{\partial x^i}{\partial \bar{x}^k}\frac{\partial \bar{x}^k}{\partial x^p}\frac{\partial x^j}{\partial \bar{x}^l}\frac{\partial \bar{x}^l}{\partial x^q} A_{ij} B^p C^q$$

$$= \frac{\partial x^i}{\partial x^p}\frac{\partial x^j}{\partial x^q} A_{ij} B^p C^q$$

$$= \delta^i_p \delta^j_q A_{ij} B^p C^q$$

$$= A_{ij} B^i C^j \qquad ...(67)$$

This proves the invariant character of $A_{ij}B^iC^j$. **Proved.**

36.11 HIGHER ORDER TENSORS

We will now generalize the definitions given for second order tensors in §9 for tensor entities of higher order.

A set of N^n quantities $A^{i_1 i_2 \ldots i_n}$ in a coordinate system x^i represents the components of a **Contravariant tensor of the order** n if the corresponding set of N^n quantities $\bar{A}^{p_1 p_2 \ldots\ldots p_n}$ in the coordinate system $\bar{x}^i$ are given by the transformation law

$$\bar{A}^{p_1 p_2 \ldots\ldots p_n} = \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}}\frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}}\cdots\frac{\partial \bar{x}^{p_n}}{\partial x^{i_n}} A^{i_1 i_2 \ldots i_n} \quad \textbf{(contravariant law)} \qquad ...(68)$$

Similarly, if we have N^n quantities $A_{i_1 i_2 \ldots\ldots i_n}$ whose transformation law is

$$\bar{A}_{q_1 q_2 \ldots\ldots q_n} = \frac{\partial x^{i_1}}{\partial \bar{x}^{q_1}}\frac{\partial x^{i_2}}{\partial \bar{x}^{q_2}}\cdots\frac{\partial x^{i_n}}{\partial \bar{x}^{q_n}} A_{i_1 i_2 \ldots\ldots i_n} \quad \textbf{(covariant law)} \qquad ...(69)$$

we call $A_{i_1 i_2 \ldots\ldots i_n}$ the components of a Covariant tensor of the order n.

Further, if we have N^{m+n} quantities

$$A^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_n}$$

whose transformation law is

$$\bar{A}^{p_1 p_2 \cdots p_m}_{q_1 q_2 \cdots q_n} = \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}}\frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}}\cdots\frac{\partial \bar{x}^{p_m}}{\partial x^{i_m}}\cdot\frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}}\frac{\partial x^{j_2}}{\partial \bar{x}^{q_2}}\cdots\frac{\partial x^{j_n}}{\partial \bar{x}^{q_n}} A^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_n}$$

(mixed tensor law) ...(70)

then we call $A^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_n}$ the components of a mixed tensor of the $(m + n)^{\text{th}}$ order, contravariant of m^{th} order and Covariant of n^{th} order, which is generally written as of the type (m, n).

Remarks:

(*i*) *The convenient way to remember the results* (68), (69) *and* (70) *is that in the right hand side expression as if 9the unbarred indices, assuming superscripts as numerator and subscripts as denominator and regarding the index i in* ($\partial \bar{x}^p / \partial x^i$) *as a subscript, cancel out and leaving the barred indices required in the left hand side.*

(*ii*) *A contravariant tensor of second order may be called a tensor of type* (2, 0) *and a covariant tensor of second order is called a tensor of type* (0, 2) *and a mixed tensor of second order is of the type* (1, 1).

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Theorem 6. ***The transformation of the tensors form a group*** **(*i.e., the law of transformation of tensors possesses* transitive property).**

Proof : Without loss of generality, we consider a mixed tensor A^i_j in a coordinate system x^i, and consider the transformation of coordinates from x^i to $\bar{x}^j$ and then x'^l. Let the corresponding 1 components of the tensor be $\bar{A}^k_l$ and A'^p_q, then

$$\bar{A}^k_l = \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^l} A^i_j \qquad ...(71)$$

and

$$A'^p_q = \frac{\partial x'^p}{\partial \bar{x}^k} \frac{\partial \bar{x}^l}{\partial x'^q} \bar{A}^k_l \qquad ...(72)$$

Combining (71) and (72), we get

$$A'^p_q = \frac{\partial x'^p}{\partial \bar{x}^k} \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial \bar{x}^l}{\partial x'^q} \frac{\partial x^j}{\partial \bar{x}^l} A^i_j$$

$$= \frac{\partial x'^p}{\partial x^i} \frac{\partial x^j}{\partial x'^q} A^i_j \qquad ...(73)$$

Equation (73) is of the same form as we get when we make direct transformation from x^i to x'^l. Hence the proposition.

Theorem 7. ***If all the components of a tensor in one coordinate system are zero at a point then they are zero at this point in every coordinate system.***

Proof : Let the components of a tensor in the coordinate system x^i be

$$A^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_n} \qquad ...(74)$$

and the corresponding components in $\bar{x}^i$ be

$$\bar{A}^{p_1 p_2 \cdots p_m}_{q_1 q_2 \cdots q_n}$$

Then by the transformation law of tensors (70), it clearly follows that if the components defined in (74) in x^i are all zero then the corresponding components defined in (75) in $\bar{x}^i$ will also be zero. Hence the proposition.

Remarks:

(*i*) *This theorem is very important in the formulation of physical laws. It immediately follows that if a tensor equation holds in one coordinate system it holds in every coordinate system.*

(*ii*) *Two tensors are said to be equal, if they are of the same rank and type and componentwise equal.*

36.12 ZERO TENSOR

Definition : A tensor whose components relatively to every coordinate system are all zero is known as zero tensor.

Notes :

(*i*) *The tensor of order zero* (*scalar or invariant*) *and zero tensor are two different concepts.*

(*ii*) *If a tensor is zero in one coordinate system it will remain zero in all other coordinate systems*.

The order of indices in a tensor is important therefore we shall study its symmetric and Anti-symmetric (skew-symmetric) properties.

Symmetric and Anti-symmetric

(skew-symmetric) Tensors

36.13 SYMMETRIC TENSORS

Definition : A tensor is called symmetric with respect to two contravariant or two covariant indices if its components remain unaltered upon interchange of the indices.

e.g. $$A_{st}^{pqr} = A_{st}^{qpr}$$

is symmetric in p and q and if

$$A_{st}^{pqr} = A_{st}^{qpr}$$

then it is said to be symmetric in s and t.

If a tensor is symmetric with respect to any two contravariant indices and also any two covariant indices then it is called **symmetric tensor**.

It may be noted that the symmetry property is defined only when the indices are of the same type.

Theorem 8. *If a tensor is symmetric with respect to two indices* (*contravariant or covariant*) *in any coordinate system it remains symmetric with respect to these two indices in any other coordinate system*.

Proof : Since only two indices are involved, there is no loss of generality if we prove the proposition for the contravariant tensor, *viz.*, $A^{ij} = A^{ji}$.

We have
$$\begin{aligned}\bar{A}^{kl} &= \frac{\partial \bar{x}^k}{\partial x^i}\frac{\partial \bar{x}^l}{\partial x^j} A^{ij} \quad \text{(due to symmetry)}\\ &= \frac{\partial \bar{x}^l}{\partial x^j}\frac{\partial \bar{x}^k}{\partial x^i} A^{ij}\\ &= \bar{A}^{lk}\end{aligned}$$

Hence the proposition.

Remark : *We do not define symmetry with respect to two indices of which one denotes contravariance and other covariance, because this type of symmetry is, usually, not preserved after a coordinate transformation. The kronecker delta is an exception, which is a mixed tensor, and possesses symmetry with respect to its two indices* $\left(\delta^i_j = \delta^j_i\right)$.

Theorem 9. *A symmetric tensor of the second order has at most* $\dfrac{N(N+1)}{2}$ *different components in a* V_N.

Proof : Let A_{ij} be a symmetric tensor of the order two. The total number of its components in the array, in a V_N.

$$\begin{matrix} A_{11}\ A_{12} \ldots A_{1N} \\ A_{21}\ A_{22} \ldots A_{2N} \\ \ldots\ \ldots\ \ldots\ \ldots \\ A_{N1}\ A_{N2} \ldots A_{NN} \end{matrix}$$

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are N^2 out of which all the N. diagonal terms will be different and the rest $(N^2 - N)$ will be equal in pairs due to symmetric property.

The number of such parts will be $\frac{N(N^2+1)}{2}$. Hence the total number of independent components

$$= N + \frac{(N^2 - N)}{2} = \frac{1}{2} N(N+1)$$

Corollary. The number of independent components in A_{ijk} which is symmetric in i and j is clearly

$$\frac{1}{2} N(N+1) \,.\, N = \frac{N^2(N+1)}{2}$$

36.14 SKEW-SYMMETRIC TENSORS

Definition : A tensor is called skew-symmetric (or antisymmetric) with respect to two contravariant or two covariant in dices if its components change sign upon interchange of the indices.

e.g.
$$A^{ijh}_{lni} = -A^{jih}_{lm}$$

is Skew-symmetric in i and j and if

$$A^{ijk}_{lni} = -A^{ijk}_{ml}$$

it is said to be skew-symmetric in l and m.

If a tensor is skew-symmetric with respect to any two contravariant indices and also any two covariant indices, then it is called **skew-symmetric tensor**.

Notes :

(*i*) *The property of skew-symmetry* (*like that of symmetry*) *is also independent of the choice of the coordinate system*.

(*ii*) *Skew-symmetry, like symmetry, cannot be defined with respect to the indices of which one denotes contravariance and the other covariance*.

(*iii*) *A skew-symmetric tensor* A^{ij} *of the second order has at most* $\frac{N(N-1)}{2}$ *different arithmetical components, as all the N diagonal terms* A^{ii} (*no summation*) *are zero in this case*.

(*iv*) In general a skew-symmetric (anti-symmetric) tensor of rank r $(< N)$ in V_N will have at most ${}^N C_r = \frac{N!}{r!(N-r)!}$ independent components.

If $r = N$ (*rank is the same as the range of indices*) *then number of independent components will be one only. The non-vanishing component of an anti-symmetric tensor of the type* $(0, N)$ *is* $\pm A_{123\ldots N}$, *according as N is odd or even*.

(*v*) *Skew-symmetric tensors of rank higher than the number of dimensions of the space are identically zero*.

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Addition, Subtraction and Multiplication of Tensors

36.15 FUNDAMENTAL ALGEBRAIC OPERATIONS WITH TENSORS

(*a*) Addition : *The sum of two or more tensors of the same rank and type* (*i.e. same number of contravariant indices and same number of covariant indices*) *is also a tensor of the same rank and type.*

$$\overline{A}_k^{ij} = \frac{\partial \overline{x}^i}{\partial x^l}\frac{\partial \overline{x}^j}{\partial x^m}\frac{\partial x^n}{\partial \overline{x}^k} A_n^{lm} \qquad ...(76)$$

$$\overline{B}_k^{ij} = \frac{\partial \overline{x}^i}{\partial x^l}\frac{\partial \overline{x}^j}{\partial x^m}\frac{\partial x^n}{\partial \overline{x}^k} B_n^{lm} \qquad ...(77)$$

Adding,

$$\left(\overline{A}_k^{ij} + \overline{B}_k^{ij}\right) = \frac{\partial \overline{x}^i}{\partial x^l}\frac{\partial \overline{x}^j}{\partial x^m}\frac{\partial x^n}{\partial \overline{x}^k}\left(A_n^{lm} + B_n^{lm}\right) \qquad ...(78)$$

This shows that $A_n^{lm} + B_n^{lm} = C_n^{lm}$ (say) is a tensor of the same rank.

Remark : It can easily be verified that the addition of tensors is commutative and associative.

(*b*) Subtraction: The difference of two tensors of the same rank and type is also a tensor of the same rank and type.

It follows immediately from (76) and (77) that

$$D_n^{lm} = A_n^{lm} - B_n^{lm}$$

is also a tensor of the same rank.

Further, it can be easily deduced from. (*a*) and (*b*) that any linear combination of tensors of the same rank and type is again a tensor of the same rank and type.

As for example, $\lambda A_n^{lm} + \mu B_n^{lm}$, where λ and μ are invariants (scalars), is a tensor of the same rank and type.

(*c*) Outer multiplication : *The product of two tensors, of any rank, is a tensor whose rank is the sum of the ranks of the given tensors.*

This process which involves ordinary multiplication of the components of the tensor is called the **outer product**. As for example, $A_k^{ij} B_m^l$ is the outer product of A_k^{ij} and B_m^l and may be denoted by C_{km}^{ijl} which is a tensor of 5th order contravariance of order 3 and covariance of order 2.

Note : *The converse of (c) is not always true, i.e. Not every tensor can be written as a product of two tensors of lower rank* (*e.g.* δ_j^i). *For this reason division of tensors is not always possible.*

The division, in the usual sense, of one tensor by another is not defined.

(*d*) Contraction : If one contravariant and one covariant index of a tensor (mixed tensor) are set equal, the result indicates that a summation over the equal indices (dummy indices) is to be taken according to the summation convention. This resulting sum is a tensor of rank two less than that of the original tensor. The process is called contraction.

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We shall now prove that a contracted tensor of the type (r, s) is of the type $(r-1, s-1)$.

Let the components of a tensor of the type (r, s) in the coordinate system x^i be $A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s}$ and in the coordinate system $\overline{x}^i$ be

$\overline{A}^{p_1 p_2 \dots p_r}_{q_1 q_2 \dots q_s}$ then,

$$\overline{A}^{p_1 p_2 \dots p_r}_{q_1 q_2 \dots q_s} = \frac{\partial \overline{x}^{p_1}}{\partial x^{i_1}} \frac{\partial \overline{x}^{p_2}}{\partial x^{i_2}} \dots \frac{\partial \overline{x}^{p_r}}{\partial x^{i_r}} \cdot \frac{\partial x^{j_1}}{\partial \overline{x}^{q_1}} \frac{\partial x^{j_2}}{\partial \overline{x}^{q_2}} \dots \frac{\partial x^{j_s}}{\partial \overline{x}^{q_s}} A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s} \qquad \dots(79)$$

Setting $p_1 = q_2$ (say), we get

$$\overline{A}^{q_2 p_2 \dots p_r}_{q_1 q_2 \dots q_s} = \delta^{j_2}_{i_1} \frac{\partial \overline{x}^{p_2}}{\partial x^{i_2}} \dots \frac{\partial \overline{x}^{p_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial \overline{x}^{q_1}} \frac{\partial x^{j_3}}{\partial \overline{x}^{q_3}} \dots \frac{\partial x^{j_s}}{\partial \overline{x}^{q_s}} A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s}$$

$$= \frac{\partial \overline{x}^{p_2}}{\partial x^{i_2}} \dots \frac{\partial \overline{x}^{p_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial \overline{x}^{q_1}} \frac{\partial x^{j_3}}{\partial \overline{x}^{q_3}} \dots \frac{\partial x^{j_s}}{\partial \overline{x}^{q_s}} A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s} \qquad \dots(80)$$

Since the free indices are $(r-1)$ in contra variance and $(s-1)$ in covariance, denoting,

$$\overline{A}^{q_2 p_2 \dots p_r}_{q_1 q_2 \dots q_s} = \overline{C}^{\alpha_1 \alpha_1 \dots \alpha_{r-1}}_{\beta_1 \beta_2 \dots \beta_{s-1}} \text{ and } A^{j_2 i_2 \dots i_r}_{j_1 j_2 \dots j_s} = C^{l_1 l_2 \dots l_{r-1}}_{m_1 m_2 \dots m_{s-1}}$$

the relation (80) may be written as

$$\overline{C}^{\alpha_1 \alpha_2 \dots \alpha_{r-1}}_{\beta_1 \beta_2 \dots \beta_{s-1}} = \frac{\partial \overline{x}^{\alpha_1}}{\partial x^{l_1}} \dots \frac{\partial \overline{x}^{\alpha_{r-1}}}{\partial x^{l_{r-1}}} \cdot \frac{\partial x^{m_1}}{\partial \overline{x}^{\beta_1}} \frac{\partial x^{m_2}}{\partial \overline{x}^{\beta_2}} \dots \frac{\partial x^{m_{s-1}}}{\partial \overline{x}^{\beta_{s-1}}} C^{l_1 l_2 \dots l_{r-1}}_{m_1 m_2 \dots m_{s-1}} \qquad \dots(81)$$

This shows that the new tensor obtained on contraction is of the type $(r-1, s-1)$.

Notes:

(*i*) *We never contract indices of the same type as the resulting sum is not necessarily a tensor.*

(*ii*) *The process of contraction reduces the order by two and may be repeatedly used, if so desired, to construct new tensors, whose order will always be non-negative.*

(*iii*) *The invariant* A^i_j *is formed by contraction from the mixed tensor* A^i_j *of order two. This justifies us in calling an invariant as a tensor of order zero.*

(*e*) Inner multiplication: *By the process of outer multiplication of two tensors (different type or mixed type) followed by a contraction, we obtain s a new tensor called an inner product of the given tensors. The process is called inner multiplication.*

As for example, given the tensors A^{ij}_k and B^p_{qr} the outer product is $A^{ij}_k B^p_{qr}$. Letting $j = q$ we obtain the inner product $A^{ij}_k B^p_{jr} = C^{ip}_{kr}$. Letting $j = q$, $i = r$ another inner product $A^{ij}_k B^p_{ji} = D^p_k$ is obtained.

Notes :

(*i*) *Inner or outer multiplication of tensors is commutative and associative.*

(*ii*) *The summation convention generally applies to two indices one of which is a superscript and the other a subscript.*

Theorem 10. ***Every tensor, which has at least two contravariant or two covariant indices, can be expressed as the sum of two tensors, one of which is symmetric and the other skew-symmetric in a pair of contravariant or covariant indices.***

Proof : Without loss of generality, let the tensor be A_k^{ij}, then we may write

$$A_k^{ij} = \frac{1}{2}\left(A_k^{ij} + A_k^{ji}\right) + \frac{1}{2}\left(A_k^{ij} - A_k^{ji}\right) \qquad ...(82)$$

Denoting, $$\frac{1}{2}\left(A_k^{ij} + A_k^{ji}\right) = B_k^{ij}$$

and $$\frac{1}{2}\left(A_k^{ij} - A_k^{ji}\right) = C_k^{ij}$$

we find, $$B_k^{ij} = B_k^{ji} \text{ and } C_k^{ij} = -C_k^{ji}$$

Hence, $$A_k^{ij} = B_k^{ij} + C_k^{ij}, \qquad ...(83)$$

in which B_k^{ij} is symmetric and C_k^{ij} is skew-symmetric.

Note : *The symmetry and anti-symmetry are sometimes shown by putting the parentheses and brackets respectively.'Here the symmetric part may be written as* $B_k^{(ij)}$. *and anti-symmetric part is* $C_k^{[ij]}$.

Example 1. If A^{rs} is skew-symmetric and B_{rs} is symmetric, prove that $A^{rs} B_{rs} = O$.

Solution: Given $A^{rs} = -A^{rs}$ and $B_{rs} = B_{sr}$.

Now changing the dummy suffixes in $A^{rs} B_{rs}$, we get

$$A^{rs}B_{rs} = A^{sr}B_{sr} = -A^{rs}B_{rs}$$

or $$2A^{rs}B_{rs} = O$$

or $$A^{rs}B_{rs} = O$$

Example 2. If $\phi = a^{ij}A^iA^j$, then we can always write $\phi = b_{ij}A^iA^j$ where b_{ij} is symmetric.

Solution: $$\phi = a^{ij}A^iA^j \qquad ...(84)$$

interchanging the dummy indices, we get

$$\phi = a^{ij}A^iA^j \qquad ...(85)$$

Adding (84) and (85)

$$2\phi = (a_{ij} + a_{ji})\, A^iA^j$$

or $$\phi - b_{ij}A^iA^j \qquad ...(86)$$

Where $$b_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) \qquad ...(87)$$

which is symmetric, *i.e.* $b_{ij} = b_{ji}$.

Example 3. If A_{ij} is a skew-symmetric tensor, then show that

$$\left(\delta_j^i\, \delta_l^k + \delta_l^i\, \delta_j^k\right) A_{ik} = 0$$

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Solution:

$$\left(\delta^i_j\ \delta^k_l + \delta^i_l\ \delta^k_j\right) A_{ik} = \delta^i_j\ \delta^k_l A_{ik} + \delta^i_l\ \delta^k_j A_{ik}$$
$$= \delta^k_l A_{ik} + \delta^k_j A_{ik}$$
$$= A_{jl} + A_{lj}$$
$$= A_{jl} + (-A_{jl}) \qquad \textbf{(skew-symmetric property)}$$
$$= 0.$$

Example 4. If a_{ij} is a symmetric covariant tensor and b_i a covariant vector which satisfy the relation

$$a_{ij}b_k + a_{jk}\ b_i + a_{ki}b_j = 0$$

prove that either $a_{ij} = 0$ *or* $b_i = 0$.

Solution: Let, $$a_{ij}b_k = A_{ijk}, \qquad ...(88)$$

then A_{ijk} is a third order covariant tensor which is symmetric with respect to the first pair of indices i and j due to the symmetric property of a_{ij} Also replacing the free indices i, j and k by j, k and i respectively on both side, we find

$$a_{jk}\ b_i = A_{jki}, \qquad ...(89)$$

is symmetric with respect to j and k and similarly

$$a_{ki}\ b_j = A_{kij}, \qquad ...(90)$$

is symmetric with respect to k and i.

Hence, A_{ijk} is a **completely symmetric tensor**.

Adding (.88) to (90) and using the given relation, we get

$$A_{ijk} + A_{jki} + A_{kij} = 0$$

or $$3\ A_{ijk} = 0,$$

or $$a_{ij}b_k = 0.$$

This implies that either $a_{ij} = 0$ or $b_k = 0$ *i.e.* $b_i = 0$.

Example 5. If the tensors a_{ij} and g_{ij} are symmetric and u^i and v^i are components of contravariant vectors satisfying the equations :

$$(a_{ij} - kg_{ij})\ u^i = 0$$

and $$(a_{ij} - k^1 g_{ij})\ v^i = 0, \qquad k \neq k^1$$

prove that $$g_{ij}\ u^i\ v^i = 0 \text{ and } a_{ij}\ u^i\ v^i = 0.$$

Solution: Given, $$a_{ij} = a_{ii},\ g_{ij} = g_{ji} \qquad ...(91)$$
$$(a_{ij} - kg_{ij})\ u^i = 0 \qquad ...(92)$$
$$(a_{ij} - k^1 g_{ij})\ v^i = 0, \qquad ...(93)$$

Taking the inner product of (92) by v^i and (93) by u^j, we

get $$a_{ij}u^i v^j - kg_{ij}u^i v^j = 0, \qquad ...(94)$$
$$a_{ij}v^i u^j - k^1 g_{ij} v^i u^j = 0. \qquad ...(95)$$

Changing the dummy indices i and j in (95) and using (91), the equation (95) may be written as

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$$a_{ij}v^j u^i - k^1 g_{ij} v^j u^i = 0. \qquad ...(96)$$

From (94) and (96), it follows

$$(k^1 - k)\, g_{ij} v^i u^j = 0,$$

or $$g_{ij} v^i u^j = 0, \text{ as } k \neq k^1$$

Substituting this in (96), we find $a_{ij} u^i v^j = 0$.

Example 6. *If* $u_{ij} \neq 0$ *are the components of a tensor of the type* (0, 2) *and if the equation* :

$$fu_{ij} + gu_{ji} = 0$$

holds, then prove that either $f = g$ *and* u_{ij} *is skew-symmetric or* $f = -g$ *and* u_{ij} *is symmetric.*

Solution: Given that $fu_{ij} + gu_{ji} = 0$. ...(1)

Changing the free indices, we may write it as

$$Fu_{ji} + gu_{ij} = 0. \qquad ...(2)$$

Adding (1) and (2), we get

$$(f + g)(u_{ij} + u_{ji}) = 0 \qquad ...(3)$$

which implies that

(*i*) either, $u_{ij} + u_{ji} = 0$, *i.e.* u_{ij} is skew-symmetric and then from (1) it follows that $f = g$,

(*ii*) or, $f = -g$ and then from (1) it follows that u_{ij} is symmetric.

Example 7. If A_{ij} is skew-symmetric, then prove that

$$\left(B_j^i\, B_l^k + B_l^i\, B_j^k\right) A_{ik} = 0$$

Solution:

$$\left(B_j^i\, B_l^k + B_l^i\, B_j^k\right) A_{ik} = B_j^i\, B_l^k A_{ik} + B_l^i\, B_j^k A_{ik}$$

$$= B_j^k B_l^i A_{ki} + B_l^i\, B_j^k A_{ik}$$ (changing the dummy indices in the first term)

$$= B_j^k B_l^i \left(A_{ki} + A_{ik}\right)$$

$$= 0.$$ (since A_{ik} is skew-symmetric)

Quotient Law of Tensors

36.16 QUOTIENT LAW

In tensor analysis it becomes sometimes necessary to ascertain whether a given entity is a tensor or not. In theory we may say that if the components of the entity obey tensor transformation laws, then it is a tensor otherwise not However, in practice this is trouble-some and a simple test is provided by a law known as Quotient law which states :

Statement : An entity whose inner product with an arbitrary tensor is a tensor, is itself a tensor.

Proof: It will suffide to set out the proof for the following particular case :

In the coordinate system x^i let $A(i, j, k)$ be the given entity. Let B_m^{ij} be an arbitrary tensor whose **inner product** with $A(i, j, k)$ is a tensor C_{mk} *i.e.*

$$A(i, j, k)\; B_m^{ij} = C_{mk}. \qquad ...(97)$$

We have to show that $A(i, j, k)$ is a tensor.

In the transformed coordinates $\bar{x}^i$, we have

$$\bar{A}(p, q, r)\, \bar{B}_n^{pq} = \bar{C}_{nr} \qquad ...(98)$$

But,
$$\bar{B}_n^{pq} = \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^q}{\partial x^j}\frac{\partial x^m}{\partial \bar{x}^n} B_m^{ij} \qquad ...(99)$$

and
$$\bar{C}_{nr} = \frac{\partial x^m}{\partial \bar{x}^n}\frac{\partial x^k}{\partial \bar{x}^r} C_{mk} \qquad ...(100)$$

Hence equation (98) may be written as

$$\bar{A}(p, q, r)\frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^q}{\partial x^j}\frac{\partial x^m}{\partial \bar{x}^n} B_m^{ij} = \frac{\partial x^m}{\partial \bar{x}^n}\frac{\partial x^k}{\partial \bar{x}^r} C_{mk}$$

$$= \frac{\partial x^m}{\partial \bar{x}^n}\frac{\partial x^k}{\partial \bar{x}^r} A(i, j, k) B_m^{ij} \quad \text{[using (97)]}$$

or
$$\frac{\partial x^m}{\partial \bar{x}^n}\left[\bar{A}(p, q, r)\frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^q}{\partial x^j} - \frac{\partial x^k}{\partial \bar{x}^r} A(i, j, k)\right] B_m^{ij} = 0$$

On inner multiplication by $\frac{\partial \bar{x}^n}{\partial x^s}$ (*i.e.* taking outer product by $\frac{\partial \bar{x}^t}{\partial x^s}$ and then contraction with $n = t$) yields

$$\frac{\partial x^m}{\partial \bar{x}^n}\left[\bar{A}(p, q, r)\frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^q}{\partial x^j} - \frac{\partial x^k}{\partial \bar{x}^r} A(i, j, k)\right] B_m^{ij} = 0$$

or
$$\left[\bar{A}(p, q, r)\frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^q}{\partial x^j} - \frac{\partial x^k}{\partial \bar{x}^r} A\,(i, j, k)\right] B_s^{ij} = 0 \qquad ...(101)$$

From this we cannot jump immediately to the conclusion that the quantity inside the parentheses vanishes. We must remember that here i and j are dummy indices which imply summation and it is the sum which is zero. However since B_s^{ij} is an arbitrary tensor we can arrange that only one of its components is non-zero. Now each component of B_s^{ij} may be chosen in turn as that one which does not vanish. Therefore the expression in brackets is identically zero.

Hence,
$$\bar{A}(p, q, r)\frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^q}{\partial x^j} = \frac{\partial x^k}{\partial \bar{x}^r} A\,(i, j, k) \qquad ...(102)$$

Again on inner multiplication with $\frac{\partial x^i}{\partial \bar{x}^m}\frac{\partial x^j}{\partial \bar{x}^n}$, we get

$$\bar{A}(p, q, r)\, \delta_m^p\, \delta_n^q = \frac{\partial x^k}{\partial \bar{x}^r}\frac{\partial x^i}{\partial \bar{x}^m}\frac{\partial x^j}{\partial \bar{x}^n} A\,(i, j, k)$$

or
$$\bar{A}(m, n, r) = \frac{\partial x^i}{\partial \bar{x}^m}\frac{\partial x^j}{\partial \bar{x}^n}\frac{\partial x^k}{\partial \bar{x}^r} A\,(i, j, k) \qquad ...(103)$$

This equation shows that $A(i, j, k)$ is a tensor of third order, which is covariant in i, j and k and therefore may be written as A_{ijk}.

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Remark : *In the above proof it is necessary that* B_m^{ij} *should be arbitrary and does not possess any symmetric or skew-symmetric properties. If it is not so, then (102) is not a logical consequence of (101).*

Example 1. *An entity A(p, q, r, s), which is a function of coordinates* x^i *transform to* $\overline{A}(i, j, k, l)$ *in another coordinate system* $\overline{x}^i$ *according to-the law*

$$\overline{A}(i, j, k, l) = \frac{\partial x^p}{\partial \overline{x}^i} \frac{\partial \overline{x}^j}{\partial x^q} \frac{\partial \overline{x}^k}{\partial x^r} \frac{\partial \overline{x}^l}{\partial x^s} A(p, q, r, s)$$

Answer the following questions :

(*i*) *Is the given entity a tensor* ?

(*ii*) *If so, give the suitable notation indicating its contravariant and covariant characters artel the rank.*

Solution: (*i*) Yes, the given entity is a tensor because it obeys the tensor transformation law.

(*ii*) The suitable notation for the given entity is A_p^{qrs} in the coordinate system x^i and $\overline{A}_i^{jkl}$ in the coordinate system $\overline{x}^i$ .This indicates that it is a mixed tensor of Order 4, contravariant of order 3 and covariant of order one, *i.e.*, of the type (3,1).

Example 2. *Use Quotient law to prove that Kronecker delta is a mixed tensor of order two.*

Solution: Let A^j be an arbitrary contravariant vector, then by an obvious property of the kronecker delta.

$$\delta_j^i A^j = A^i,$$

which is again a tensor (A^i is contravariant tensor of order one).

Hence, by Quotient law δ_j^i is a tensor.

Moreover δ_j^i has two indices i and j and with its product with A^j the summation is carried out over j *i.e.* it should be covariant in j and since the result is the contravariant vector A^i it should be contravariant in i. Thus δ_j^i is a mixed tensor.

Example 3. *If* A^i *and* B^i *are arbitrary contravariant vectors and* $C_{jj}A^iB^j$ *is an invariant, show that* C_{jj} *is a covariant tensor of the second order.*

Solution: Since $C_{ij}A^iB^j$ is an invariant, we have

$$C_{ij} A^i B^j = \overline{C}_{pq} \overline{A}^p \, \overline{B}^q \, . \quad ...(104)$$

Further, A^i and B^i are contravariant vectors, therefore

$$\overline{A}^p = \frac{\partial \overline{x}^p}{\partial x^i} A^i \quad ...(105)$$

and
$$\overline{B}^q = \frac{\partial \overline{x}^q}{\partial x^j} B^j \quad ...(106)$$

Substituting these in (104), we get ′

$$\left(C_{ij} - \overline{C}_{pq} \frac{\partial \overline{x}^p}{\partial x^i} \frac{\partial \overline{x}^q}{\partial x^j} \right) A^i B^j = 0 \quad ...(107)$$

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But A^i, B^j W being arbitrary vectors, it follows that A^iB^j is an arbitrary tensor and therefore

$$C_{ij} = \overline{C}_{pq} \frac{\partial \overline{x}^p}{\partial x^i} \frac{\partial \overline{x}^q}{\partial x^j} \qquad ...(108)$$

which is the law of transformation of a contravariant tensor of the order two. Hence the result.

Example 4. *If A^i is an arbitrary contravariant vector and $C_{ij}A^iA^j$ is an invariant, show that $C_{ij} + C_{ji}$ is a covariant tensor of the second order.*

Solution: Proceeding as in Example 16, the equation (107) in the present case may be written as

$$\left(C_{ij} - \overline{C}_{pq} \frac{\partial \overline{x}^p}{\partial x^i} \frac{\partial \overline{x}^q}{\partial x^j} \right) A^i A^j = 0 \qquad ...(109)$$

This quadratic form, vanishes for arbitrary A^i, but we cannot jump to the conclusion that the quantity inside the parent-heses vanishes because A^iA^j is not arbitrary but a symmetric tensor. We must remember that in the form $b_{ij}A^iA^j$ the coefficient of the product A^1 A^2 is mixed up with the coefficient of A^2 A^1; it is in fact b_{12} + b_2 Thus interchanging the dummy indices i and j, and adding the two results, we can deduce only that

$$C_{ij} + C_{ji} = \overline{C}_{pq} \frac{\partial \overline{x}^p}{\partial x^i} \frac{\partial \overline{x}^q}{\partial x^j} + \overline{C}_{qp} \frac{\partial \overline{x}^p}{\partial x^j} \frac{\partial \overline{x}^q}{\partial x^i} \qquad ...(110)$$

The trick now is to interchange the dummies p and q to the last term; this gives

$$\left(C_{ij} + C_{ji} \right) = \left(\overline{C}_{pq} + \overline{C}_{qp} \right) \frac{\partial \overline{x}^p}{\partial x^i} \frac{\partial \overline{x}^q}{\partial x^j} \qquad ...(111)$$

establishing the tensor character of $(C_{ij} + C_{ji})$ as a covariant tensor of the order two.

36.17 RELATIVE TENSOR

If the components of an entity

$$A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s}$$

transform according to the equation

$$\overline{A}^{p_1 p_2 \dots p_r}_{q_1 q_2 \dots q_s} = \left| \frac{\partial x}{\partial \overline{x}} \right|^w \frac{\partial \overline{x}^{p_1}}{\partial x^{i_1}} \frac{\partial \overline{x}^{p_2}}{\partial x^{i_2}} \dots \frac{\partial \overline{x}^{p_r}}{\partial x^{i_r}} \cdot \frac{\partial x^{j_1}}{\partial \overline{x}^{q_1}} \dots \frac{\partial x^{j_s}}{\partial \overline{x}^{q_s}} A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s} \qquad ...(112)$$

then the given entity is Called a relative tensor of weight w, where $\left| \frac{\partial x}{\partial \overline{x}} \right|$ is the **Jacobian of transformation**. If $w = 0$, the entity is called **absolute tensor** or simply **tensor**. If $w = 1$, the relative tensor is called the tensor density.

Notes :

(*i*) *The algebraic operations, multiplication, addition subtraction of* **relative tensor** *are same as those of absolute tensors.*

(*ii*) *The outer product of two* **relative tensors** *is itself a relative tensor of rank and weight equal to the sum of the ranks and the sum of weights of the given relative tensors respectively.*

(*iii*) *Unless or otherwise stated we are dealing with absolute tensors in this text.*

Conjugate Tensors

36.18 CONJUGATE TENSORS (OR RECIPROCAL TENSORS)

Lemma 1. *Consider a symmetric covariant tensor of the second order A_{ij} whose determinant $|A_{ij}| \neq 0$. Let $G_{(A)}(i, j)$* denote the cofactor of A_{ij} in the determinant $|A_{ij}|$ and let*

$$B^{ij} = \frac{G_{(A)}(i, j)}{|A_{ij}|}. \qquad ...(113)$$

We shall now prove that B^{ij} represents the components of a symmetric contravariant tensor of the order two.

We have labelled the indices i and j in B^{ij} as contra variance indices in anticipation of the result.

Proof. Since A_{ij} is symmetric, $|A_{ij}|$ is symmetric which implies that $G_{(A)}(i, j)$ is symmetric and therefore B^{ij} is symmetric.

From the properties of determinants we have the following two results

$$A_{ij}\, G_{(A)}(i, j) = |A_{ij}| \qquad ...(114)$$

$$A_{ij}\, G_{(A)}(i, k) = 0, j \neq k. \qquad ...(115)$$

Hence, using (113) we may write the above two results by a single equation

$$A_{ij} B^{ik} = \delta_j^k \qquad ...(116)$$

Although δ_j^k is a tensor and A_{ij} is a tensor, we cannot apply Quotient law to establish tensor character of B^{ik} to this equation because A_{ij} is not arbitrary. It is a symmetric covariant tensor.

Let C^i be a chosen arbitrary contravariant vector, then

$$C^i A_{ij} = D_i, \qquad ...(117)$$

so that D_i is an arbitrary covariant vector, because the above equations can be solved uniquely as $|A_{ij}| \neq 0$. Consequently.

$$C^j A_{ij} B^{ik} = D_i B^{ik}$$

or $$C^j \delta_j^k = D_i B^{ik}$$

or $$C^k = D_i B^{ik} \qquad ...(118)$$

We can now apply the Quotient law to equation (118) and see that B^{ik} is a contravariant tensor of the second order.

Lemma 2. Let us now form another tensor E_{ij} from B^{ij} by the same process as defined in **Lemma 1**, *i.e.*

$$E_{ij} = \frac{G_{(B)}(i, j).}{|B^{ij}|} \qquad ...(119)$$

Since $|A_{ij}|.|B^{ij}| = 1$ and $|A_{ij}| \neq 0$, it follows that $|B^{ij}| \neq 0$. We shall now prove that $E_{ij} = A_{ij}$.

Proof : By the theory of determinants

$$E_{ij} B^{ik} = \delta^k_j. \quad ...(120)$$

Inner multiplication by A_{kl} yields,

$$E_{ij} B^{ik} A_{kl} = \delta^k_j A_{kl} \text{ using (116),. } E_{ij}\delta^i_l = A_{jl}$$

$$E_{lj} = A_{jl} = A_{lj} \quad ...(121)$$

Hence the proposition.

Thus we see that, by virtue of (116), the relation between the tensors A_{ij} and B^{ij} is of reciprocal nature. We, therefore, give the following definition of conjugate tensors (or reciprocal tensors) :

Definition : Two second order symmetric tensors A_{ij} and B^{ij}. one covariant and the other contravariant, are said to be conjugate (or reciprocal) tensors if they satisfy the equation

$$A_{ij} B^{ik} = \delta^k_j \text{ and } |A_{ij}|.|B^{ij}| \neq 0.$$

Note : if the tensor A_{ij} is given, Lemma 1 describes the process by which its conjugate B^{ij} can be determined. Similarly, Lemma 2, describes the process to determine A_{ij} when B^{ij} is given.

Example 1. *If A_{ij} is a symmetric covariant tensor of the order two and B^{ij} is formed by dividing the cofactor of A_{ij} in the determinant $|A_{ij}| = a$ (say) by $|A_{ij}|$ itself, show that :*

(*i*) $|B^{ij}| = 1/a$ and (*ii*) $A_{ij}B^{ij} = N$.

Solution: By the theory of determinants

$$A_{ij} B^{ik} = \delta^k_j \quad ...(122)$$

(*i*) Now, $|A_{ij}|.|B^{ik}| = |\delta^k_j|$ or $a|B^{ik}| = 1$

$$|B^{ik}| = \frac{1}{a} \quad ...(123)$$

(*ii*) Again, from (122) identifying *j* and *k*, we get

$$A_{ij} B^{ij} = \delta^j_j = N. \quad ...(124)$$

Example 2. *If $A_{ij} = 0$ for $i \neq j$, show that the conjugate tensor $B^{ij} = 0$ for $i \neq j$ and $B^{ii} = \frac{1}{A_{ii}}$ (no summation).*

Solution: We have $A_{ij} B^{ik} = \delta^k_j$ ...(125)

(i) Let $k \neq j$, then

$$\begin{aligned} 0 &= A_{ij}B^{ik} \\ &= A_{1j}B^{1k} + A_{2j}B^{2k} + ... + A_{jj}B^{jk} + ... + A_{Nj}B^{Nk} \\ &= 0 + 0 + ... + A_{jj}B^{jk} + ... + 0 \\ &= A_{jj}B^{jk} \text{ (No summation over } j). \end{aligned}$$

But, $A_{jj} \neq 0$ (No summation over *j*).

Hence $B^{jk} = 0, j \neq k$ i.e., $B^{ij} = 0, i \neq j$ ...(126)

(*ii*) Let $k = j$, then from (125)

$$\begin{aligned} 1 &= A_{ij}B^{ij} \\ &= A_{i1}B^{i1} + A_{i2}B^{i2} + ...+ A_{ii}B^{ii} + ...+ A_{iN}B^{iN} \\ &= 0 + 0 + ...+ A_{ii}B^{ii} + ...+ 0 \\ &= A_{ii}B^{ii}. \text{ (No summation over } i\text{).} \end{aligned}$$

But, $A_{ii} \neq 0$ (No summation over i).

Hence $$B^{ii} = \frac{1}{A^{ii}} \text{ (No summation)} \qquad ...(127)$$

Note : We shall use the results of Example 19, hereafter, as the standard results in the succeeding chapters.

Example 3. *If A^{ij} and A_{ij} are reciprocal symmetric tensors and if u_i are components of a covariant vector, show that $A_{ij}u^iu^j = A^{ij}u_iu_j$, where $u^i = A^{i\alpha}u_\alpha$.*

Solution: Since $u^i = A^{i\alpha}u_\alpha$

taking the inner multiplication by A_{ir}, we get

$$A_{ir}u^i = A_{ir}A^{i\alpha}u_\alpha = \delta_r^\alpha u_\alpha = u_r. \qquad ...(128)$$

Now with the help of (128), we have

$$\begin{aligned} A^{ij}u_iu_j &= A^{ij}(A_{ki}u^k)\ (A_{lj}u^l) = \delta_k^j u^k A_{lj}u^l \\ &= A_{lj}u^lu^j \\ &= A_{ij}u^iu^j. \end{aligned}$$

Example 4. *If the relation $B_{ijk}u^iu^ju^k = 0$ holds for any arbitrary contravariant vector u^i, show that*

$$B_{ijk} + B_{jki} + B_{kij} + B_{jik} + B_{ikj} + B_{kji} = 0.$$

Solution: We have $B_{ijk}u^iu^ju^k = 0.$

Also, be changing the dummy indices, we get successively

$$B_{jki}u^ju^ku^i = 0, \qquad B_{kij}u^ku^iu^j = 0, \ B_{jik}u^ju^iu^k = 0,$$
$$B_{ikj}u^iu^ku^j = 0, \qquad B_{kji}u^ku^ju^i = 0.$$

In this way all the permutations are exhausted. On addition, these six equations give

$$(B_{ijk} + B_{jki} + B_{kij} + B_{jik} + B_{ikj} + B_{kji})u^iu^ju^k = 0.$$

This implies for arbitrary u^i, i.e. for not necessarily zero contravariant vector u^i,

$$B_{ijk} + B_{jki} + B_{kij} + B_{jik} + B_{ikj} + B_{kji} = 0.$$

Example 5. *If the tensor B_{ijk} is symmetric in i and j and the relation $B_{ijk}A^iA^jA^k = 0$ holds for any arbitrary contravariant vector A^i, show that* $B_{ijk} + B_{jki} + B_{kij} = 0.$

Solution: The symmetry of the tensor B_{ijk} in i and j implies the symmetry in its first two free indices, i.e.

$$B_{ijk} = B_{jik}, \qquad B_{jki} = B_{kji}, \qquad \text{and } B_{kij} = B_{ikj} \qquad ...(1)$$

Further, changing the dummy indices and taking all the permutation as explained in Example 21, we conclude from $B_{ijk}A^iA^jA^k = 0$, that

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$$B_{ijk} + B_{jki} + B_{kij} + B_{jik} + B_{ikj} + B_{kji} = 0 \quad ...(2)$$

From (1) and (2), we get the desired result

$$B_{ijk} + B_{jki} + B_{kij} = 0.$$ **Proved.**

Example 6. If a tensor A_{ijkl} is symmetric in i and j and anti-symmetric in j and l, show that $A_{ijkl} = 0$.

Solution: The given conditions imply that the tensor A_{ijkl} is symmetric in the first and second indices and anti-symmetric in second and fourth indices. Using the symmetric and anti-symmetric properties, we may write

$$A_{ijkl} = A_{jikl} = -A_{jlki} \quad ...(1)$$

Also, $$A_{ijkl} = -A_{ilkj} = -A_{likj} = A_{ljki} = A_{jlki} \quad ...(2)$$

Adding (1) and (2), we get

$$2A_{ijkl} = -A_{jlki} + A_{jlki} = 0$$

Hence, $$A_{ijkl} = 0.$$ **Proved.**

Example 7. If $A^i_{jk} B^{jk} = C^i$ is a contravariant vector and B^{jk} is an anti-symmetric tensor, then show that $A^i_{jk} + A^i_{kj}$ is a tensor.

Solution: Given that $$A^i_{jk} B^{jk} = C^i. \quad ...(1)$$

In the transformed coordinates , we have

$$\bar{A}^p_{qr} \bar{B}^{qr} = \bar{C}^P. \quad ...(2)$$

But $\bar{B}^{qr}$ and $\bar{C}^P$ are tensors, therefore

$$\bar{B}^{qr} = \frac{\partial \bar{x}^q}{\partial x^j} \frac{\partial \bar{x}^r}{\partial x^k} B^{jk}, \quad ...(3)$$

and $$\bar{C}^p = \frac{\partial \bar{x}^p}{\partial x^i} C^i. \quad ...(4)$$

Hence equation (2) may be written as

$$\bar{A}^p_{qr} \frac{\partial \bar{x}^q}{\partial x^j} \frac{\partial \bar{x}^r}{\partial x^k} B^{jk} = \frac{\partial \bar{x}^p}{\partial x^i} C^i = \frac{\partial \bar{x}^p}{\partial x^i} A^i_{jk} B^{jk} \quad \text{[using (1)]}$$

or . $$\left[\bar{A}^p_{qr} \frac{\partial \bar{x}^q}{\partial x^j} \frac{\partial \bar{x}^r}{\partial x^k} - \frac{\partial \bar{x}^p}{\partial x^i} A^i_{jk} \right] B^{jk} = 0 . \quad ...(5)$$

Since B^{jk} is an anti-symmetric tensor (not arbitrary) from this we cannot jump to the conclusion that the quantity inside parentheses vanishes. Interchanging the dummy indices j and k, we get

. $$\left[\bar{A}^p_{qr} \frac{\partial \bar{x}^q}{\partial x^k} \frac{\partial \bar{x}^r}{\partial x^j} - \frac{\partial \bar{x}^p}{\partial x^i} A^i_{kj} \right] B^{kj} = 0. \quad ...(6)$$

Now changing the dummy indices q and r within the parentheses and writing $B^{kj} = - B^{jk}$, equation (6) may be written as

$$\left[\bar{A}^p_{rq} \frac{\partial \bar{x}^r}{\partial x^k} \frac{\partial \bar{x}^q}{\partial x^j} - \frac{\partial \bar{x}^p}{\partial x^i} A^i_{kj} \right] B^{kj} = 0. \quad ...(7)$$

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Adding (5) and (7), we get

$$\left[\left(\bar{A}^p_{qr} + \bar{A}^p_{rq}\right)\frac{\partial \bar{x}^q}{\partial x^j}\frac{\partial \bar{x}^r}{\partial x^k} - \left(A^i_{jk} + A^i_{kj}\right)\frac{\partial \bar{x}^p}{\partial x^i}\right]B^{jk} = 0. \qquad ...(8)$$

This implies that (because the result still holds good when j and k are interchanged),

$$\left(\bar{A}^p_{qr} + \bar{A}^p_{rq}\right)\frac{\partial \bar{x}^q}{\partial x^j}\frac{\partial \bar{x}^r}{\partial x^k} = \left(A^i_{jk} + A^i_{kj}\right)\frac{\partial \bar{x}^p}{\partial x^i}$$

or $$\left(A^i_{jk} + A^i_{kj}\right) = \frac{\partial x^i}{\partial \bar{x}^p}\frac{\partial \bar{x}^q}{\partial x^j}\frac{\partial \bar{x}^r}{\partial x^k}\left(\bar{A}^p_{qr} + \bar{A}^p_{rq}\right) \qquad ...(9)$$

establishing the tensor character of $\left(A^i_{jk} + A^i_{kj}\right)$ as a mixed tensor of the type (1,2).

Proved.

36.19 ASSOCIATION OF A SKEW SYMMETRIC TENSORS OF ORDER TWO AND VECTORS

We associate the skew symmetric tensor of order two.

$$a_{ij} = \in_{ijk} a_k \qquad ...(1)$$

The tensor a_{ij} is skew symmetric for

$$a_{ji} = \in_{jlk} a_k = -\in_{ijk} a_k = -a_{ij}$$

The relation (1) is equivalent to statements

$a_{23} = a_1$, $a_{32} = -a_1$; $a_{31} = a_2$, $a_{13} = -a_2$; $a_{12} = a_3$, $a_{21} = -a_3$; $a_{11} = 0$, $a_{22} = 0$; $a_{33} = 0$.

On the inner multiplication with $\hat{I}_{ijm}$ we obtain from (1)

$$\in_{ijm} a_{ij} = \in_{ijm} \in_{ijk} a_k = 2\delta_{mk} a_k = 2a_m \text{ when } k = m$$

$$\in_{ijk} \in_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

$$\in_{ijk} \in_{pjk} = \delta_{ip}\delta_{jj} - \delta_{jp}\delta_{ij} = 3\delta_{ip} - \delta_{ip} = 2\delta_{ip}$$

Hence $a_m = \dfrac{1}{2} \in_{ijm} a_{ij}$

This shows that association is one-one.

EXERCISE 36.1

1. Write down the laws of transformation for the tensors A^{ij}_k and B^{ij}_{klm}
2. Evaluate: (*a*) $\delta^i_j \delta^i_k$ (*b*) $\delta^i_j \delta^i_k \delta^i_l$
3. Show that the velocity of a fluid at any point is a contravariant tensor of rank one.
4. If A^i_j is a mixed tensor of rank two, show that A^i_j is also a tensor.
5. In an N-dimensional space, how many different expressions are represented by $A^{ip}_j B^k_{ir} C^{it}_{sk}$? When each expression is written out explicitly, how many terms does it contain?

6. If A_k^{ij} and B_r^{pq} are tensors, show that $A_i^{ij} B_i^{pi}$ is not a tensor.

7. If A_{kl}^{ij} is a tensor, show that A_{kl}^{ii} and A_{kk}^{ij} are not tensors.

8. If A_{lm}^{ijk} is a tensor, show that $A_{jk}^{ijk}, A_{kj}^{ijk}, A_{lm}^{ljm}$ and A_{lm}^{lmk} are contravariant vectors.

9. Show that any covariant or contravariant tensor of the second rank can be expressed as the sum of a symmetric tensor and an antisymmetric tensor of the same rank and type.

10. If *a*, *b*, *c* are three-dimensional vectors, show that their scalar triple product can be written as (**a** × **b**). c = $\varepsilon_{ijk} a_i b_j c_k$, where a_i, b_i, c_i are the cartesian components of ***a***, ***b***, ***c*** respectively, and summation convention is used.

11. If a_i is any vector, show that $e_{ijk} a_j a_k = 0$.

    If A^i are the components of an absolute contravariant tensor of rank one, show that $\dfrac{\partial A_i}{\partial x_j}$ are the components of a mixed tensor.

12. If A^{ji} and A_{ij} are reciprocal symmetric tensors and x_i are the components of a covariant tensor of rank one, show that $A_{ij} x^i x^j, = A^{ij} x_i x_j$ where $x^i = A^{i\alpha} x_\alpha$.

13. If the components of a tensor are zero in one co-ordinate system, then prove that the components are zero in all co-ordinate systems.

14. Show that the expression *A(i,j,k)* is a tensor if its inner product with an arbitrary tensor B_k^{jl} is a tensor.

15. A^{ij} is a contravariant tensor and B_i a covariant tensor. Show that $A^{ij} B_k$ is a tensor of rank three, but $A^{ij} B_j$ is a tensor of rank one.

16. If g_{ij} denotes the components of a covariant tensor of rank two, show that the product $g_{ij}\, dx^i\, dx^j$ is an invariant scalar.

ANSWERS

1. $\bar{A}_k^{ij} = \dfrac{\partial \bar{x}^i}{\partial x^p} \dfrac{\partial \bar{x}^j}{\partial x^q} \dfrac{\partial x^r}{\partial \bar{x}^k} A_r^{pq}, \bar{B}_{klm}^{ij} = \dfrac{\partial \bar{x}^i}{\partial x^p} \dfrac{\partial \bar{x}^j}{\partial x^q} \dfrac{\partial x^r}{\partial \bar{x}^k} \dfrac{\partial x^s}{\partial \bar{x}^l} \dfrac{\partial x^t}{\partial \bar{x}^m} B_{rst}^{rq}$ **2.** **(a)** $\delta_k^i\,(b)\,\delta_j^i$

Metric Tensor

36.20 EUCLIDEAN SPACE OF THREE-DIMENSIONS

In the familiar Euclidean space of three-dimensions in rectangular cartesian coordinates the distance *ds* between two neighbouring points (x^1, x^2, x^3) and $(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$ is given by

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = dx^i\, dx^i, \ (i = 1, 2, 3). \quad ...(1)$$

The distance ds is also called the line-element.

The formula (1) is called the metric of the Euclidean space of three dimensions and it contains within itself all the basic elements of the geometry of a rectangular space of three dimensions.

In equation (1) the coefficients of the squares of dx^1, dx^2 and dx^3 are equal to unity and no terms like $dx^1 dx^2$, etc. occur. These properties are, however, due to the use of orthogonal cartesian coordinates and disappear if any other coordinate systems are used.

If instead of rectangular cartesian coordinates we take the coordinates of the points in curvilinear coordinates (e.g. cylindrical or spherical polar coordinates) such as $\left(x'^1, x'^2, x'^3\right)$ then x^i are functions of x'^i and dx^i are linear homogeneous functions of the dx'^i given by (c.f. §3)

$$dx^i = \frac{\partial x^i}{\partial x'^m} dx'^m \,(i, m = 1, 2, 3) \qquad ...(2)$$

When we substitute these linear functions in (1) we get a homogeneous quadratic expression in dx'^i viz.,

$$ds^2 = \left(\frac{\partial x^i}{\partial x'm}\frac{\partial x^i}{\partial x'^n}\right) dx'^m dx'^n \quad \text{(Summation over } i\text{)} \qquad ...(3)$$

which may be written as

$$ds^2 = g'_{mn} dx'^m dx'^n \quad (m,\ n = 1,\ 2,\ 3) \qquad ...(4)$$

where $$g'_{mn} = \frac{\partial x^i}{\partial x'^m}\frac{\partial x^i}{\partial x'^n} \text{(Summation over } i\text{)} \qquad ...(5)$$

No matter what curvilinear coordinates are used, the distance between two given points has the same value, i.e. ds (or ds^2) is an invariant.

The differential expression on the right hand side of (2.4) which represents ds^2 may be called the metric form or fundamental form of the space under consideration. It may also be called the square of the line element.

Motivated by this, the idea of distance was extended by Riemann, originator of tensor calculus, to a space of N-dimensions.

Metric and Fundamental Tensors

36.21 RIEMANNIAN SPACE, METRIC TENSOR

Definition : If the square of the line element ds between two neighbouring points, whose coordinates in a V_N are x^i and $x^i + dx^i$, is given by the quadratic differential form

$$ds^2 = g_{ij} dx^i dx^j, \qquad ...(6)$$

where g_{ij} are functions of x^i and $g = |g_{ij}| \neq 0$, the space is said to be Riemannian space.

In addition to this we postulate that the line element ds is independent of the coordinate system i.e. ds^2 is an invariant. It follows from (6) (see theorem 1) that g_{ij} is a symmetric covariant tensor of the order two. It is called the fundamental covariant tensor or metric tensor of the Riemannian space. The quadratic differential form $g_{ij}dx^i dx^j$ is called the Riemannian metric or simply the metric of the space.

Theorem 1. The fundamental tensor g_{ij} is a covariant symmetric tensor of the order two.

Proof : Since $g_{ij}dx^i dx^j$ is an invariant, we have

$$g_{ij} dx^i dx^j = \bar{g}_{pq} d\bar{x}^p d\bar{x}^q \qquad ...(7)$$

$$= \bar{g}_{pq} \frac{\partial \bar{x}^p}{\partial x^i} dx^i \frac{\partial \bar{x}^q}{\partial x^j} dx^j$$

or $$\left(g_{ij} - \bar{g}_{pq}\frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^q}{\partial x^j}\right)dx^i dx^j = 0. \qquad \ldots(8)$$

we deduce from equation (8) that

$$g_{ij} + g_{ji} = \left(\bar{g}_{pq} + \bar{g}_{qp}\right)\frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^q}{\partial x^j} \qquad \ldots(9)$$

establishing that $(g_{ij} + g_{ji})$ is a covariant tensor of the second order.

We can write

$$g_{ij} = \frac{1}{2}\left(g_{ij} + g_{ji}\right) + \frac{1}{2}\left(g_{ij} - g_{ji}\right) \qquad \ldots(10)$$

Then $$g_{ij}dx^i dx^j = \frac{1}{2}\left(g_{ij} + g_{ji}\right)dx^i dx^j + \frac{1}{2}\left(g_{ij} - g_{ji}\right)dx^i dx^j \qquad \ldots(11)$$

also, on interchanging the dummy indices on the R.H.S., we get

$$g_{ij}dx^i dx^j = \frac{1}{2}\left(g_{ji} + g_{ij}\right)dx^i dx^j + \frac{1}{2}\left(g_{ji} - g_{ij}\right)dx^j dx^i \qquad \ldots(12)$$

Adding equations (11) and (12), we get

$$2g_{ij}dx^i dx^j = \left(g_{ij} + g_{ji}\right)dx^i dx^j \qquad \ldots(13)$$

This equation implies that g_{ij} is symmetric. Thus combining the two conclusions that $(g_{ij} + g_{ji})$ is a covariant tensor of the second order and g_{ij} is symmetric, we conclude that $2g_{ij}$, or g_{ij} is a symmetric covariant tensor of the second order.

Note : If we compare (1) and (6) we find that in a three dimensional Euclidean space, referred to a system of rectangular axes, all the components of fundamental tensor are zero except $g_{11} = g_{22} = g_{33} = 1$

We shall call a N-dimensional space as Euclidean space of N-dimensions if its metric is

$$ds^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \ldots\ldots + \left(dx^N\right)^2 \qquad \ldots(14)$$

i.e. $g_{ij} = 0,\ i \neq j$ and $g_{ii} = 1$ (no summation).

Indicator

It is implied that the metric of a **Euclidean space** is positive definite. *i.e.*

$$ds^2 \geq 0. \qquad \ldots(15)$$

In special theory of relativity the metric of the four dimensional space (space-time) is given by

$$ds^2 = -\left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2 + c^2\left(dx^4\right)^2 \qquad \ldots(16)$$

where c is the velocity of light and x^4 is the time coordinate. This metric is not positive definite, we see that ds^2 is positive when x^1, x^2, x^3 are constants along the curves, it is zero when, say, x^2 and x^3 are constants and $x^1 = cx^4$, and negative when x^4 is constant.

Thus, in general, for some displacements dx^i the form ds^2 may be positive and for others it may be zero or negative. If $ds^2 = 0$, for dx^i not all zero, i.e. the two points are not coincident the displacement is called a **null** displacement. A curve along which the displacement $g_{ij}dx^i dx^j$ is null despite the fact that the two points are not coincident is called a **null curve.** For any displacement dx^i which is not null, we introduce an **indicator** e, which is +1 or –1, so as to make ds^2 always positive, *i.e.*

$$ds^2 = e\, g_{ij}\, dx^i\, dx^j \qquad \ldots(17)$$

36.22 CONJUGATE METRIC TENSOR

Since g_{ij} is a symmetric covariant tensor of the second order and $g = |g_{ij}| \neq 0$, we can define

$$g^{ij} = \frac{G(i, j)}{g}, \qquad \ldots 18)$$

Where $G\,(i,j)$ is the expression formed by the cofactor of g_{ij} in the determinant $|g_{ij}|$.

If follows that g^{ij} is a symmetric contravariant tensor of the second order and is said to be the conjugate of g_{ij}, *i.e.* **conjugate metric tensor.** It is also called the **fundamental contravariant tensor.** Hence the fundamental covariant tensor g_{ij} and fundmental contravariant tensor g^{ij}, being conjugate, are related to each other by the equation.

$$g_{ij}\, g^{ik} = \delta_j^k \qquad \ldots(19)$$

36.23 METRIC TENSOR IN CARTESIAN COORDINATES

Show that the metric of a Euclidean space, referred to Cartesian coordinate is given by

$$ds^2 = dx^2 + dy^2 + dz^2$$

Here we have

$$ds^2 = g_{ij} dx^i dx^j = \bar{g}_{pq} d\bar{x}^p d\bar{x}^q \qquad \ldots(1)$$

In Cartesian coordinates

$$\bar{x} = x,\ \bar{x}^2 = y,\ \bar{x}^3 = z, \qquad \ldots(2)$$

By covariant law

$$\bar{g}_{pq} = \frac{\partial x^i}{\partial \bar{x}^p} \cdot \frac{\partial x^j}{\partial \bar{x}^q} g_{ij}$$

$$\therefore\ \bar{g}_n = \frac{\partial x^i}{\partial \bar{x}^1} \cdot \frac{\partial x^j}{\partial \bar{x}^1} g_{ij} = \left(\frac{\partial x}{\partial \bar{x}^1}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^1}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^1}\right)^2 g_{33}$$

$$= \left(\frac{\partial x}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2$$

$$= 1 + 0 + 0 = 1$$

Similarly, $\bar{g}_{22} = 1,\ \bar{g}_{33} = 1$

and $g_{12} = g_{21} = g_{13} = g_{31} = g_{23} = g_{32} = 0$

$$\therefore \quad ds^2 = \bar{g}_{11}\left(dx^{-1}\right)^2 + \bar{g}_{22}\left(dx^{-2}\right)^2 + \bar{g}_{33}\left(dx^{-3}\right)^2$$

$$= dx^2 + dy^2 + \mathrm{d}z^2$$

The metric tensor $\bar{g}_{pq}$ in cartesian coordinates, is

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$$\bar{g}_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad ...(3)$$

and the conjugate metric tensor $\bar{g}_{pq}$, which is the inverse of the matrix(3) is

$$\bar{g}^{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

36.24 METRIC TENSOR IN CYLINDRICAL COORDINATES

Example. Show that the metric of a Euclidean space, referred to cylindrical coordinates is given by

$$ds^2 = (dr)^2 + (rd\theta)^2 + (dz)^2.$$

Determine its metric tensor and conjugate metric tensor.

Solution: We have

$$ds^2 = g_{ij}\, dx^i\, dx^j = \bar{g}^{pq}\, d\bar{x}^p d\bar{x}^q \qquad ...(1)$$

In cylindrical coordinates

$$\bar{x}^1 = r,\ \bar{x}^2 = \theta,\ \bar{x}^3 = z;\ x = r\cos\theta,\ y = r\sin\theta,\ z = z \qquad ...(2)$$

and $\bar{g}_{pq} = ?$

By covariant law

$$\bar{g}_{pq} = \frac{\partial x^i}{\partial \bar{x}^p}\frac{\partial x^j}{\partial \bar{x}^q} g_{ij}.$$

Therefore,

$$\bar{g}_{11} = \frac{\partial x^i}{\partial \bar{x}^1}\frac{\partial x^j}{\partial \bar{x}^1} g_{ij}$$

$$= \left(\frac{\partial x^1}{\partial \bar{x}^1}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^1}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^1}\right)^2 g_{33}$$

$$= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2$$

$$= \cos^2\theta + \sin^2\theta + 0 = 1. \qquad ...(3)$$

$$\bar{g}_{22} = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2$$

$$= r^2\sin^2\theta + r^2\cos^2\theta + 0 = r^2 \qquad ...(4)$$

$$\bar{g}_{33} = \left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2$$

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$$= 0+0+1=1 \quad \text{...(5)}$$

$$\overline{g}_{12} = \frac{\partial x^i}{\partial \overline{x}^1}\frac{\partial x^j}{\partial \overline{x}^2} g_{ij}$$

$$= \left(\frac{\partial x^1}{\partial \overline{x}^1}\right)\left(\frac{\partial x^1}{\partial \overline{x}^2}\right) g_{11} + \left(\frac{\partial x^2}{\partial \overline{x}^1}\right)\left(\frac{\partial x^2}{\partial \overline{x}^2}\right) g_{22} + \left(\frac{\partial x^3}{\partial \overline{x}^1}\right)\left(\frac{\partial x^3}{\partial \overline{x}^2}\right) g_{33}$$

$$= \left(\frac{\partial x}{\partial r}\right)\left(\frac{\partial x}{\partial \theta}\right) + \left(\frac{\partial y}{\partial r}\right)\left(\frac{\partial y}{\partial \theta}\right) + \left(\frac{\partial z}{\partial r}\right)\left(\frac{\partial z}{\partial \theta}\right)$$

$$= \text{-r}\cos\theta\sin\theta + \text{r}\sin\theta\cos\theta + 0$$

$$= 0 \quad \text{...(6)}$$

Similarly,

$\overline{g}_{13} = \overline{g}_{23} = 0,$ and due to symmetric property $\overline{g}_{21} = \overline{g}_{31} = \overline{g}_{32} = 0.$

Hence,

$$ds^2 = \overline{g}_{11}\left(d\overline{x}^1\right)^2 + \overline{g}_{22}\left(d\overline{x}^2\right)^2 + \overline{g}_{33}\left(d\overline{x}^3\right)^2$$

$$= (dr)^2 + (rd\theta)^2 + (dz)^2 \quad \text{...(7)}$$

The metric tensor in cylindrical coordinates is

$$\overline{g}_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{...(8)}$$

Clearly, $g = \left|\overline{g}_{pq}\right| = r^2$ ...(9)

and the conjugate metric tensor $\overline{g}^{pq}$, which is the inverse of the matrix (8), is

$$\overline{g}^{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ans.

36.25 METRIC TENSOR IN SPHERICAL COORDINATES

Example. Show that the metric of a Euclidean space, referred to spherical coordinates is given by

$$ds^2 = (dr)^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2$$

Determine its metric tensor and conjugate metric tensor.

Solution: We have

$$ds^2 = g_{ij}dx^i dx^j = \overline{g}_{pq} d\overline{x}^p d\overline{x}^q \quad \text{...(1)}$$

In spherical polar coordinates

$$\overline{x}^1 = r,\ \overline{x}^2 = \theta,\ \overline{x}^3 = \phi$$

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta \quad \text{...(2)}$$

We have to find $\bar{g}_{pq}$.

By covariant law

$$\bar{g}_{pq} = \frac{\partial x^i}{\partial \bar{x}^p}\frac{\partial x^j}{\partial \bar{x}^q} g_{ij}$$

Therefore,

$$\bar{g}_{11} = \frac{\partial x^4}{\partial \bar{x}^1}\frac{\partial x^j}{\partial \bar{x}^1} g_{ij}$$

$$= \left(\frac{\partial x^1}{\partial \bar{x}^1}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^1}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^1}\right)^2 g_{33}$$

$$= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2$$

$$= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1 \qquad ...(3)$$

$$g_{22} = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2$$

$$= r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta = r^2 \qquad ....(4)$$

$$\bar{g}_{33} = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2$$

$$= r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\, s\cos^2\phi + 0$$

$$= r^2\sin^2\theta. \qquad ...(5)$$

$$\bar{g}_{12} = \frac{\partial x^i}{\partial x^{-1}}\frac{\partial x^j}{\partial x^{-2}} g_{ij}$$

$$= \frac{\partial x^1}{\partial x^{-1}}\frac{\partial x^1}{\partial x^{-2}} g_{11} + \frac{\partial x^2}{\partial x^{-1}}\frac{\partial x^2}{\partial x^{-2}} g_{22} + \frac{\partial x^3}{\partial x^{-1}}\frac{\partial x^3}{\partial x^{-2}} g_{33}$$

$$= \left(\frac{\partial x}{\partial r}\right)\left(\frac{\partial x}{\partial \theta}\right) + \left(\frac{\partial y}{\partial r}\right)\left(\frac{\partial y}{\partial \theta}\right) + \left(\frac{\partial z}{\partial r}\right)\left(\frac{\partial z}{\partial \theta}\right)$$

$$= (\sin\theta\cos\phi)(r\cos\theta\cos\phi) + (\sin\theta\sin\phi)(r\cos\theta\sin\phi) + (\cos\theta)(-r\sin\theta)$$

$$= r\sin\theta\cos\theta\left(\cos^2\phi + \sin^2\phi\right) - r\sin\theta\cos\theta$$

$$= 0. \qquad (6)$$

Similarly,

$$\bar{g}_{13} = \bar{g}_{23} = 0 \text{ and by symmetric property } \bar{g}_{21} = \bar{g}_{31} = \bar{g}_{32} = 0$$

Hence,

$$ds^2 = \bar{g}_{11}\left(d\bar{x}^1\right)^2 + \bar{g}_{22}\left(d\bar{x}^2\right)^2 + \bar{g}_{33}\left(d\bar{x}^3\right)^2$$

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$$= (dr)^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2 \qquad \ldots(7)$$

The metric tensor $\bar{g}_{pq}$ in spherical polar coordinates is therefore given by

$$\bar{g}_{p_q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix} \qquad \ldots(8)$$

Clearly,

$$g = \left|\bar{g}_{pq}\right| = r^4 \sin^2\theta \qquad \ldots(9)$$

and the conjugate metric tensor $\bar{g}^{pq}$, which is the inverse of the matrix (8), is

$$\bar{g}^{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/r^2\sin^2\theta \end{bmatrix}$$

Ans.

Example 1. If the metric of a V_3 is given by

$$ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(dx^3)^2 - 6(dx^1)(dx^2) + 4(dx^2)(dx^3),$$

Find (i) g and (ii) g^{ij}

Solution: Comparing the given metric, with the metric

$$ds^2 = g_{ij}dx^i dx^j, \quad (i, j = 1,2,3)$$

We find

$$g_{11} = 5, \; g_{12} = g_{21} = -3$$
$$g_{22} = 3, \; g_{23} = g_{32} = 2$$
$$g_{33} = 4, \; g_{13} = g_{31} = 0.$$

Hence,

$$g_{ij} = \begin{bmatrix} 5 & -3 & 0 \\ -3 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix} \qquad \ldots(1)$$

and

$$g = \left|g_{ij}\right| = 4 \qquad \ldots(2)$$

To get the conjugate of g_{ij}, *i.e.* the inverse of the matrix equation (1)

$G(1,1) = 8$, $\quad G(1,2) = G(2,1) = 12$, $\quad G(2,3) = G(3.2) = -10$,

$G(2,2) = 20$, $\quad G(3,1) = G(1,3) = -6$,

$G(3,3) = 6$

Since,

$$g^{ij} = \frac{G(i.j)}{g}, \text{ we obtain}$$

$g^{11} = 2$, $g^{12} = g^{21} = 3.$

$g^{22} = 5$, $g^{23} = g^{32} = -\frac{5}{2}$

$g^{33} = \frac{3}{2}$, $g^{31} = g^{13} = -\frac{3}{2}$

Hence,

$$g^{ij} = \begin{bmatrix} 2 & 3 & -\frac{3}{2} \\ 3 & 5 & -\frac{5}{2} \\ -\frac{3}{2} & -\frac{5}{2} & \frac{3}{2} \end{bmatrix}$$ **Ans.**

Example 2. *Show that*

(i) $(g_{hj}\, g_{ik} - g_{hk}\, g_{ij})\, g^{hj} = (N - 1)\, g_{ik}$

(ii) $\frac{\partial \phi}{\partial x^j}\, (g_{hk}\, g_{il} - g_{hl}\, g_{ik})\, g^{hj} = \frac{\partial \phi}{\partial x^k} g_{il} - \frac{\partial \phi}{\partial x^1}\, g_{ik}$; ϕ is a scalar.

(iii) *If* g_{ij} *and* a_{ij} *are components of two symmetric tensors and* $g_{ij}\, a_{kl} - g_{il}\, a_{kj} + g_{kj}\, a_{il} - g_{kl}\, a_{ij} = 0$

($i, j, k, l = 1, 2,\ldots, N$), show that $a_{ij} = \alpha\, g_{ij}$, where α is a scalar.

Solution: We have

$$g_{ij}\, g^{ik} = g^k_j$$

(i)
$$\begin{aligned}(g_{hj}\, g_{ik} - g_{hk}\, g_{ij})\, g^{hj} &= g_{hj}\, g^{hj} - g^{hj}\, g_{hk}\, g_{ij} \\ &= Ng_{ik} - \delta^j_k\, g_{ij} \\ &= Ng_{ik} - g_{ik} \\ &= (N - 1)\, g_{ik}\end{aligned}$$

(ii)
$$\begin{aligned}\frac{\partial \phi}{\partial x^j}\, (g_{hk}\, g_{il} - g_{hl}\, g_{ik})\, g^{hj} &= \frac{\partial \phi}{\partial x^j} g_{hk} g^{hj} g_{il} - \frac{\partial \phi}{\partial xj} g_{hl}\, g^{hj}\, g_{ik} \\ &= \frac{\partial \phi}{\partial xj} \delta^j_k g_{il} - \frac{\partial \phi}{\partial xj} \delta^j_l g_{ik} \\ &= \frac{\partial \phi}{\partial x^k} g_{il} - \frac{\partial \phi}{\partial x^l} g_{ik}\end{aligned}$$

(iii) Since g^{ij} is the conjugate tensor of the given tensor g_{ij}, taking the inner product of the given tensor equation by g^{ij}, we have

$$g^{ij}\, (g_{ij}\, a_{kl} - g_{il}\, a_{jk} + g_{jk}\, a_{il} - g_{kl}\, a_{ij}) = 0$$

or $$g^{ij}\, g_{ij}\, a_{kl} - g^{ij}\, g_{il}\, a_{jk} + g^{ij}\, g_{jk}\, a_{il} - g^{ij}\, a_{ij}\, g_{kl} = 0$$

or $$Na_{kl} - \delta^j_l\, a_{jk} + \delta^i_k\, a_{il} - \beta\, g_{kl} = 0$$

[Since $g^{ij}\, g_{il} = \delta^i_1$ and $g^{ij}\, a_{ij} = a$ scalar (= β, say)]

or $$Na_{kl} - a_{lk} + a_{kl} - \beta\, g_{kl} = 0$$

or $$Na_{kl} = \beta\, g_{kl},\qquad [a_{lk} = a_{kl}]$$

or $$a_{kl} = \frac{\beta}{N} g_{kl} = \alpha g_{kl},\qquad [\alpha = \beta/N,\ a \text{ scalar}]$$

or $$a_{ij} = \alpha g_{ij}$$ **Proved**

Example 3. *Show that*

(*i*) $g^{ij}\, g^{kl}\, dg_{ik} = -dg^{jl}$ (*ii*) $g_{ij}\, g_{kl}\, dg^{ik} = -\, dg_{jl}$

Solution: We have

(*i*) $$g^{ij}\, g_{ik} = \delta^j_k \qquad ...(1)$$

On differentiation

$$g^{ij}\, dg_{ik} + g_{ik}\, dg^{ij} = 0$$

or $$g^{ij}\, dg_{ik} = -g_{ik}\, dg^{ij} = 0 \qquad ...(2)$$

Taking inner product of (2) by g^{kl}, we get

$$g^{ij}\, g^{kl}\, dg_{ik} = -g^{kl}\, g_{ik}\, dg^{ij}$$
$$= -\delta^l_i\, dg^{ij}$$
$$= -dg^{lj}$$
$$= -dg^{jl} \text{ (by symmetric property)}$$ **Proved.**

(*ii*) Relation (1) may be written as

$$g^{ik}\, g_{ij} = \delta^k_j$$

On differentiation

$$g_{ij}\, dg^{ik} + g^{ik}\, dg_{ij} = 0$$

or $$g_{ij}\, dg^{ik} = -g^{ik}\, dg_{ij} \qquad ...(3)$$

Taking inner product of (3) by g_{kl}, we get

$$g_{ij}\, g_{kl}\, dg^{ik} = -g^{ik}\, g_{kl}\, dg_{ij}$$
$$= -\delta^i_l\, dg_{ij}$$
$$= -dg_{lj}$$
$$= -dg_{jl}$$ **Proved**

Associate Tensors

36.26 ASSOCIATE VECTORS

Definition: *The associate vector of a contra variant vector A^j s defined as the inner product of the fundaments tensor g_{ij} and A^j and denoted by A_i.*

Thus $$A_i = g_{ij} A^j \qquad ...(1)$$

The covariant vector A_i is the associate vector of the contravariant vector A^j and the process is called the lowering of the superscript.

In a similar manner we may define the associate vector of the covariant vector B_i by

$$B^i = g^{ij}\, B_j. \qquad ...(2)$$

The contravariant vector B^i is the associate vector of the covariant vector B_j and the process is called the raising of the subscript.

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Theorem 2. The relation between a vector and its associate is reciprocal.

Proof: Let A^i be a given contravariant vector and A_j be its associate, then

$$A_j = g_{ij} A^i \quad ...(3)$$

The associate of the covariant vector A_j is , by definition (say B^i)

$$B^i = A_j g^{ij}$$ [substituting (3), after

$$= g^{ij} g_{kj} A^k$$ changing the dummy suffix i]

$$= \delta^i_k A^k = A^i \quad ...(4)$$

This show that the associate of the associate is the vector itself and thus establishes the reciprocal character. **Proved.**

36.27 ASSOCIATE TENSORS

The process of raising and lowering the indices can be performed on tensors of higher order. From the tensor A^{ijk}_{lm} we can from **associate tensors like**

$$A^{jk}_{rim} = g_{ir} A^{ijk}_{lm}$$

$$A^{...k}_{rslm} = g_{ir} g_{js} A^{ijk}_{lm}$$

$$A^{rijk}_{..m} = g^{rl} A^{ijk}_{lm}$$

The dot notation is used to indicate the indices which have been raised or lowered.

The dots may be omitted when there is no scope of confusion, e.g., we may write $A^{pq} = g^{ip} g^{jq} A_{ij}$. It may be noted that an associate tensor of g_{ij} is

$$g^{pi} g^{qj} g_{ij}$$

$$= g^{pi} \delta^q_i$$

$$= g^{pq}$$

This shows that the **fundamental tensors** g_{ij} and g^{ij}, besides being conjugate are also **associate tensors**. However, any second.

Order tensor and its associate, like A_{ij} and A^{ij} may not be conjugate as a rude.

Example. *Show that* $A^{\alpha\beta} dg_{\alpha\beta} = - A_{\alpha\beta} dg^{\alpha\beta}$

Solution. We know that:- $dg_{jt} = -g_{ij} g_{kl} dg^{ik}$

Taking the inner product by A^{jl}, we get

$$A^{jl} dg_{jl} = - g_{ij} A^{jl} g_{kl} dg^{ik}$$

$$= - A^{l}_{i} g_{kl} dg^{ik}$$

$$= - A_{ik} dg^{ik},$$

Changing the dummy indices, we get the required result

$$A^{\alpha\beta} dg_{\alpha\beta} = -A_{\alpha\beta} dg^{\alpha\beta}$$ **Proved**

36.28 MAGNITUDE OF A VECTOR

The fundamental quantities required for any geometrical measurement are length and angle. These can be defined and calculated with the help of the metric tensor and that is why the metric tensor often refers to as geometry of the space.

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Definition: *The* **magnitude** *of a contravariant vector A^i, which is usually denoted by A, is defined by the square of length or norm of vector A^i as*

$$(A)^2 = e_{(A)}\, g_{ij}\, A^i A^j, \qquad ...(5)$$

or

$$(A)^2 = e_{(A)}\, A_j\, A^j \qquad ...(6)$$

where $e_{(A)}$ is the indicator +1 or –1, which make A real. The magnitude A is a invariant. In Euclidean space V_3 referred to rectangular cartesian coordinates, there is no difference between contravariant and covariant vectors and $e_{(A)} = +1$, the relation (2.56) reduces to the familiar definition of the magnitude of a vector, viz.,

$$(A)^2 = (A_1)^2 + (A_2)^2 + (A_3)^2.$$

Similarly, the magnitude B of the covariant vector B_I is defined by

$$(B)^2 = e_{(B)}\, g^{ij}\, B_i B_j \qquad ...(7)$$

or

$$(B)^2 = e_{(B)}\, B^j B_j \qquad ...(8)$$

Unit Vector: A vector whose magnitude is unity is called a unit vector.

It may be noted that

$$ds^2 = eg_{ij}\, dx^i dx^j$$

or

$$1 = eg_{ij}\left(\frac{dx^i}{ds}\right)\left(\frac{dx^j}{ds}\right) \qquad ...(9)$$

This show that (dx^i/ds) is a *unit contravariant vector*. It is a unit tangent vector to the curve in V_N.

Null Vector: *A vector whose magnitude is zero is called a null vector.*

For example the tangent vector to a null curve is a null vetor.

Note: *The indicator $e_{(A)}$ may be dropped, if it is obvious that $(A)^2$ is positive.*

36.29 ANGLE BETWEEN TWO VECTORS

In familiar vector algebra the scalar product of two vectors A and B is defined as

$$A.B = |A|\,|B| \cos\theta \qquad ...(10)$$

where θ is the angle between A and B.

$$\text{Hence, } \cos\theta = \frac{A.\text{B}}{|A||B|} \qquad ...(11)$$

Motivated by this the angle between two vector A^i and B^i in Riemannian space is defined by

$$\cos\theta = \frac{A^i B_i}{\sqrt{e_{(A)} A^l A_l \left(e_{(B)} B^m B_m\right)}}$$

$$= \frac{g_{ij} A^i B^j}{\sqrt{\left(e_{(A)} g_{lp} A^l A^p\right)\left(e_{(B)} 8_{mq} B^m B^q\right)}} \qquad ...(12)$$

If the two vectors A^i and B^i are unit vectors, then

$$\cos\theta = A^i B_i = g_{ij}\, A^i\, B^j = A_i\, B^j. \qquad ...(13)$$

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Orthogonal Vectors: *Two vectors are said to be* ***orthogonal*** *if the angle between them is a right angle, i.e.* $\cos\theta = 0$. *Hence it follows from (12) that the necessary and sufficient condition for orthogonality of two vector* A^i *and* B^i *is*

$$g_{ij} A^i B^i = 0 \qquad ...(14)$$

or

$$A^i B_i = 0 \qquad ...(15)$$

Note: *We do not define the angle between two vectors, when one or both of them happens to be null vector. However, (14) is still taken as the definition of orthogonality of two null vectors. It may be noted that for a null vector* dx^i

$$g_{ij}\, dx^i dx^j = 0. \qquad ...(16)$$

This show that the null vector is self-orthogonal.

Theorem 3. The angle between two unit vectors A^i and B^i , in a V_N, is defined by

$$\cos\theta = g_{ij} A^i B^j.$$

Show that $|\cos\theta \leq 1$, if the metric of the Riemannian space V_N is positive define.

Proof: If the metric of the Riemannian space is positive definite then the magnitude of the vetor $\lambda A^i + \mu B^i$ is greater then or equal to zero for all real values of λ and μ, i.e.,

$$g_{ij}\,(\lambda A^i + \mu B^i)\,(\lambda A^j + \mu B^i\,) \geq 0, \qquad ...(17)$$

for all real value of λ and μ.

Hence, $g_{ij}\,(\lambda^2 A^i A^j + \lambda\mu B^i A^j + \lambda\mu A^i B^j + \mu^2 B^i B^i) \geq 0$

or

$$\lambda^2 + \lambda\mu\cos\theta + \lambda\mu\cos\theta + \mu^2 \geq 0$$

or

$$(\lambda + \mu\cos\theta)^2 + \mu^2(1 - \cos^2\theta) \geq 0 \qquad ...(18)$$

Since this is true for all real value of λ and μ, it follow that

$$1 - \cos^2\theta \geq 0$$

i.e.

$$|\cos\theta| \leq \qquad ...(19)$$

Hence the proposition.

Note: If the metric is not positive then the angle between two real unit vectors need not be real.

Example 1. *If* X_{ij} *are components of a symmetric covariant tensor and* u^i, v^i *are unit vectors orthogonal to* w^i *and satisfying the relations:*

$$(X_{ij} - \omega g_{ij})\, u^i + \rho w_j = 0$$

$$(X_{ij} - \omega' g_{ij})\, v^i + \rho' w_j = 0$$

where $\omega \neq \omega'$, *prove that* u^i *and* v^i *are orthogonal, and that*

$$X_{ij}\, u^i v^i = 0.$$

Solution: Since the unit vectors u^i, v^i are orthogonal to w^i we have

$$u^i w_i = 0, \qquad ...(1)$$

$$v^i w_i = 0. \qquad ...(2)$$

Taking the inner product of the given relation

$$(X_{ij} - \omega g_{ij})\, u^i + \rho w_j = 0,$$

With v^i and using (2), we get

$$(X_{ij} - \omega g_{ij})\, u^i v^i = 0. \qquad ...(3)$$

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Similarly, the inner product of the second relation

$$(X_{ij} - \omega' g_{ij})\, v^i + \rho' w_j = 0$$

with u^j and using (1), we get

$$(X_{ij} - \omega' g_{ij})\, v^i u^j = 0. \qquad ...(4)$$

Since X_{ij} and g_{ij} are symmetric tensors, interchanging the suffixes *I* and *j* in (4), we get

$$(X_{ij} - \omega' g_{ij})\, v^j u^i = 0. \qquad ...(5)$$

Subtracting (5) from (3), we find

$$(\omega - \omega')\, g_{ij}\, u^i\, v^j = 0$$

This implies

$$g_{ij}\, u^i\, v^j = 0; \text{ as } \omega = \omega' \qquad ...(6)$$

i.e. u^i and v^i are orthogonal vectors.

Further, from (5) and (6) we conclude

$$X_{ij}\, u^i\, v^j = 0.$$ **Proved.**

Example 2. *In a three-dimensional coordinate system show that the angles between the coordinate curves are given by*

$$\cos\theta_{12} = \frac{g_{12}}{\sqrt{g_{11}g_{22}}}, \cos\theta_{13} = \frac{g_{13}}{\sqrt{g_{11}g_{33}}}, \cos\theta_{23} = \frac{g_{23}}{\sqrt{g_{22}g_{33}}}$$

Solution: Along the x^1 coordinate curve, x^2 = cont. and x^3 = cont. Therefore,

$$ds^2 = g_{11}(dx^1)^2, dx^2 = 0,\ dx^3 = 0.$$

or

$$\frac{dx^1}{ds} = \frac{1}{\sqrt{g_{11}}} \qquad ...(1)$$

Thus a unit tangent vector, which is a contravariant vector along the x^1 – curve has the components $\left(\frac{1}{\sqrt{g_{11}}}, 0, 0\right)$ and if we denote it by $A^r{}_1$, then

$$A_1^r = \frac{1}{\sqrt{g_{11}}}\delta_1^r \qquad ...(2)$$

where r = 1, 2, 3.

Similarly, the components of the unit tangent vector along the x^2–curve are given by

$$A_2^r = \frac{1}{\sqrt{g_{22}}}\delta_2^r \qquad ...(3)$$

and along the x^3–curve by

$$A_3^r = \frac{1}{\sqrt{g_{33}}}\delta_3^r \qquad ...(4)$$

Now, the angle between the coordinate curves x^1 and x^2 is the angle their unit tangent vectors (2) and (3):

Hence,

$$\cos\theta_{12} = g_{pq}\, A_1^p\, A_2^q$$

$$= g_{pq}\, \delta^p{}_1\, \delta^q{}_2\, \frac{1}{\sqrt{g_{11}g_{22}}} = \frac{g_{12}}{\sqrt{g_{11}g_{22}}} \qquad ...(5)$$

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Similarly,

$$\cos\theta_{13} = \frac{g_{13}}{\sqrt{g_{11}g_{22}}} \text{ and } \cos\theta_{23} = \frac{g_{23}}{\sqrt{g_{22}g_{22}}} \qquad ...(6)$$

Example 3. *In an orthogonal coordinate system V_3 show that*

(*i*) $g_{12} = g_{23} = g_{31} = 0$ (*ii*) $g_{11} = \frac{1}{g^{11}}\, g_{22} = \frac{1}{g^{22}}$ and $g_{33} = \frac{1}{g^{33}}$

Solution: (*i*) In the orthogonal coordinate system

$$\theta_{12} = \theta_{23} = \theta_{13} = 90°$$

Therefore, from (5) and (6) it follows that

$$g_{12} = 0,\ g_{13} = 0 \text{ and } g_{23} = 0 \qquad ...(1)$$

(*ii*) We know that

$$\theta g_{ij}\, g^{jk} = \delta_i^k \qquad ...(2)$$

Let $k = i = 1$, then

$$g_{1j}\, g^{j1} = \delta^1{}_1$$

or $$g_{11}\, g^{11} + g_{12}\, g^{21} + g_{13}\, g^{31} = 1$$

or $$g_{11}\, g^{11} + 0 + 0 = 1, \qquad \text{[using (1)]}$$

Therefore, $$g_{11} = \frac{1}{g^{11}} \qquad ...(3)$$

In a similar manner, by taking $k = i = 2$ and $k = i = 3$ respectively, we get

$$g_{22} = \frac{1}{g^{22}} \text{ and } g_{33} = \frac{1}{g^{33}} \qquad ...(4)$$

Example 4. Show that the angle θ between the vectors A^i and B^i is given by

$$\sin^2\theta = \frac{\left(e_{(A)}e_{(g)}g_{hi}g_{jk} - g_{hk}g_{ij}\right)A^h A^i B^j B^k}{e_{(A)}e_{(B)}g_{hi}g_{jk}A^h A^i B^j B^k}$$

Solution: We have, be definition

$$\cos\theta = \frac{g_{ij}A^i B^j}{\sqrt{e_{(A)}g_{hi}A^h A^i}\sqrt{e_{(B)}g_{jk}B^j B^k}} \qquad ...(1)$$

Therefore,

$$\cos^2\theta = \frac{\left(g_{ij}A^i B^j\right)\left(g_{hk}A^h B^k\right)}{\left(e_{(A)}g_{hi}A^h A^i\right)\left(e_{(B)}g_{jk}B^j B^k\right)}$$

where it is kept in mind that the dummy suffix, in multiplication, should not be repeated more than twice.

Hence, $$\sin^2\theta = 1 - \cos^2\theta$$

$$= 1 - \frac{g_{ij}g_{hk}A^i B^j A^h B^k}{e_{(A)}e_{(B)}g_{hj}g_{jk}A^h A^i B^j B^k}$$

$$= \frac{\left(e_{(A)}e_{(B)}g_{hj}g_{jk} - g_{ij}g_{hk}\right)A^i B^j A^h B^k}{e_{(A)}e_{(B)}g_{hj}g_{jk}A^h A^i B^j B^k}$$ **Proved.**

Example 5. *If* A^i and B^i are orthogonal unit vectors, show that

$$(g_{hj}\, g_{ik} - g_{hk}\, g_{ij})\, A^h\, B^i A^j B^k = 1$$

Solution: Since A^i and B^i are orthogonal unit vectors, we have

$$g_{ij}\, A^i B^j = 0 \text{ and } g_{hj}\, A^h\, A^i = 1,$$

$$g_{jk}\, B^j\, B^k = 1$$

Now, $(g_{hj}\, g_{ik} - g_{hk}\, g_{ij})\, A^h B^i A^j B^k$

$$= g_{hj} A^h A^j\, g_{ik}\, B^i\, B^k - g_{hk}\, A^h\, B^k\, g_{ij}\, B^i\, A^j$$

$$= (1)\,(1) - (0)\,(0)$$

$$= 1$$ **Proved.**

Example 6. *Prove that (1,0, 0, 0) and* $\left(\sqrt{2}, 0, 0, \sqrt{3}/c\right)$ *and unit vectors in the* V_4 *with the metric*

$$ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + c^2\,(dx^4)^2$$

Show also that the angle between these vectors is not real.

Solution: Let, $A^i = (1, 0, 0, 0)$ and $B^i = \left(\sqrt{2}, 0, 0, \sqrt{3}/c\right)$

Also for he metric

$$ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + c^2\,(dx^4)^2$$

$$g_{11} = -1,\ g_{22} = -1,\ g_{33} = -1,\ g_{44} = c^2$$

and $g_{ij} = 0,\ i \neq j$

Now, $(A)^2 = e_{(A)}\, g_{ij}\, A^i\, A^j = -g_{11}\, A^1\, A^1 = 1$

and $(B)^2 = e_{(B)}\, g_{ij}\, B^i\, B^j$

$$= e_{(B)}\,\{\, g_{11}\, B^1\, B^1 + g_{44}\, B^4\, B^4\}$$

$$= e_{(B)}\left\{-2 + c^2\,\frac{3}{c^2}\right\}$$

$$= (+1)\,(-2 + 3) = 1$$

Hence, A^i and B^i are unit vectors.

Further, $\cos\theta = g_{ij}\, A^i\, B^j$

$= g_{11}\, A^1\, B^1$, other terms being zero by virtue

$= -\sqrt{2}$ of the given values of A^i, B^i and g_{ij}

i.e. $|\cos\theta| = \sqrt{2} > 1$

This show that the angle θ is not real. **Proved**

Principal Directions

36.30 PRINCIPAL DIRECTIONS FOR A SYMMETRIC COVARIANT TENSOR OF THE SECOND ORDER

Let A_{ij} be the components of a symmetric covariant tensor of the second order. Since A_{ij} can be written as a square matrix; we consider the determinantal equation

$$| A_{ij} - \lambda g_{ij} | = 0, \qquad ...(1)$$

which is of degree N in λ.

By the covariant law, we have

$$A_{ij} = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} \bar{A}_{pq}$$

and
$$g_{ij} = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} \bar{g}_{pq}$$

Hence, in a new coordinate system $\bar{x}^i$, the equation (1) transform to

$$\left| \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} \left(\bar{A}_{pq} - \lambda \bar{g}_{pq} \right) \right| = 0$$

or
$$\left| \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} \right| \left| \bar{A}_{pq} - \lambda \bar{g}_{pq} \right| = 0$$

or
$$\left| \bar{A}_{pq} - \lambda \bar{g}_{pq} \right| = 0 \qquad ...(2)$$

Since,
$$J = \left| \frac{\partial \bar{x}^p}{\partial x^i} \right| \left| \frac{\partial \bar{x}^q}{\partial x^j} \right| \neq 0 \qquad ...(3)$$

The equations (1) and (2) are of the same form and hence the N-roots $\lambda_{(k)}$ ($k = 1, 2, \ldots, N$) of this invariants. The parentheses in k emphasise that it has no tonsorial significance.

Let $\lambda_{(k)}$ is a simple root (not repeated) of the equation (1) then the equations

$$(A_{ij} - \lambda_{(k)} g_{ij}) L^i_{(k)} = 0, \qquad ...(4)$$

which are N in number, determine the values of N-components $L^i_{(k)}$. We shall now show that $L^i_{(k)}$ are the components of a contravariant vector.

Since the tensor $(A_{ij} - \lambda_{(k)} g_{ij})$ is not arbitrary, we cannot apply Quotient law to establish the tensor character of $L^i_{(k)}$. Therefore, changing to the coordinate system $\bar{x}^i$ the equation (4) may be written as

$$\left(\bar{A}_{pq} - \lambda_{(k)} \bar{g}_{pq} \right) \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} L^i_{(k)} = 0 \qquad ...(5)$$

Taking inner multiplication by $\dfrac{\partial x^j}{\partial \bar{x}^r}$ yields

$$\left(\bar{A}_{pq} - \lambda_{(k)} \bar{g}_{pq} \right) \delta^q_r \frac{\partial \bar{x}^p}{\partial x^i} L^i_{(k)} = 0$$

or
$$\left(\bar{A}_{pq} - \lambda_{(k)} \bar{g}_{pr} \right) \frac{\partial \bar{x}^p}{\partial x^i} L^i_{(k)} = 0 \qquad ...(6)$$

These equations, which are N in number, determine the values of the N components of the entity $\frac{\partial \bar{x}^p}{\partial x^i} L^i_{(k)}$ which we represent by $\bar{L}^p_{(k)}$ in the coordinate system $\bar{x}^i$.

Thus,
$$\bar{L}^p_{(k)} = \frac{\partial \bar{x}^p}{\partial x^i} L^i_{(k)} \qquad ...(7)$$

This shows the $L^i_{(k)}$ are the components of a contravaiant vector.

The equation (4) implies that without loss of generality the components $L^i_{(k)}$ .may be taken as the components of a unit vector.

Hence,
$$e_{(k)}\, g_{ij}\, L^i_{(k)}\, L^j_{(k)} = 1$$
or
$$g_{ij}\, L^i_{(k)}\, L^j_{(k)} = e_{(k)} \qquad ...(8)$$

where $e_{(k)}$ is the indicator corresponding to the vector $L^i_{(k)}$.

Let $\lambda_{(M)}$ is another simple root of the equation (1), which is not $\lambda_{(k)}$, i.e.

$$\lambda_{(M)} \neq \lambda_{(k)} \qquad ...(9)$$

Then the components of the corresponding contravariant unit vector $L^i_{(M)}$ are given by the equation
$$(A_{ij} - \lambda_{(M)}\, g_{ij})\, L^i_{(M)} = 0 \qquad ...(10)$$

In this case
$$g_{ij}\, L^i_{(M)}\, L^j_{(M)} = e_{(M)} \qquad ...(11)$$

Now, take the inner multiplication of (2.86) by $L^j_{(M)}$ and that of (10) by $L^j_{(k)}$ we find
$$A_{ij}\, L^i_{(k)}\, L^j_{(M)} - \lambda_{(k)}\, L^i_{(k)}\, L^j_{(M)} = 0 \qquad ...(12)$$
$$A_{ij} L^i_{(M)} L^j_{(k)} - \lambda_{(M)} g_{ij} L^i_{(M)} L^j_{(k)} = 0 \qquad ...(13)$$

Since, A_{ij} and g_{ij} are both symmetric the equation (13) may be written as
$$A_{ji} L^i_{(M)} L^j_{(k)} - \lambda_{(M)} g_{ji} L^i_{(M)} L^j_{(k)} = 0,$$

Then changing the dummy suffixes i and j, we get
$$A_{ij} L^j_{(M)} L^i_{(k)} - \lambda_{(M)} g_{ij} L^j_{(M)} L^i_{(k)} = 0. \qquad ...(14)$$

Subtracting (14) from (12), we get
$$\left[\lambda_{(M)} - \lambda_{(\mathrm{k})}\right]\left[g_{ij} L^i_{(k)} L^j_{(\mathrm{M})}\right] = 0. \qquad ...(15)$$

Since, $\lambda_{(M)} \neq \lambda_{(\mathrm{k})}$ we have
$$g_{ij} L^i_{(k)} L^j_{(\mathrm{M})} = 0. \qquad ...(16)$$

This shows that the two unit vectors $L^i_{(k)}$ and $L^i_{(\mathrm{M})}$ corresponding to the two different roots, are orthogonal.

Thus we conclude that if all the N roots of the equation (1) are real and distinct, then the corresponding N unit contravariant vectors, determined by the covariant symmetric tensor A_{ij} are **mutually orthogonal**.

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The directions of these N unit vectors at a point, which are mutually orthogonal, are called the **principal directions** determined by the covariant symmetric tensor A_{ij}. The condition for the existence of the real principal directions is that the roots of (3) are real, which will be satisfied if the metric of the space is positive definite.

In Euclidean space of N-dimensions, the metric of the space is given by (previous section) and the components of the fundamental tensor g_{ij} form the $N \times N$ unit matrix, *i.e.* $[g_{ij}] = I$. ...(17)

Therefore the roots of the equation (1) in this case are **eigen values** of the matrix A_{ij} and the principal directions are the directions of the **eigen vectors.**

If $A_{ij} = lg_{ij}$ at a point, then the principal directions are indeterminate at that point. If $A_{ij} = lg_{ij}$ at all points of a space V_N, the space is said to be **homogeneous** with respect to the symmetric tensor A_{ij}.

Example 1. *Show that the principal directions at a point for the symmetric tensor A_{ij} correspond to the maximum and minimum values of λ defined by*

$$\lambda = \frac{A_{ij} L^i L^j}{g_{lm} L^l L^m}$$

Solution: We are given that

$$\lambda = \frac{A_{ij} L^i L^j}{g_{lm} L^l L^m} \qquad ...(1)$$

or $$A_{ij} L^i L^j = \lambda g_{lm} L^l L^m$$

$$\left(A_{ij} - \lambda g_{ij}\right) L^i L^j = 0. \quad (l, m \text{ are dummy suffixes}) \qquad ...(2)$$

For maximum or minimum values of λ, we have

$$\frac{\partial \lambda}{\partial L^j} = 0. \qquad ...(3)$$

Differentiating (2) with respect to L^j and using (3), we get

$$2\left(A_{ij} - \lambda g_{ij}\right) L^i = 0$$

or $$\left(A_{ij} - \lambda g_{ij}\right) L^i = 0. \qquad ...(4)$$

Equation (4) implies that the directions L^i determined by A_{ij} are the principal directions, which is the required result.

Example 2. *Prove that*

$$A_{ij} L^i_{(k)} L^j_{(k)} = e_{(k)} \lambda_{(k)} \qquad \text{[No summation over } (k)\text{]}$$

and $$A_{ij} L^i_{(k)} L^j_{(m)} = 0.$$

Solution: From equation (4), we conclude that

$$A_{ij} L^i_{(k)} L^j_{(k)} = \lambda_{(k)} g_{ij} L^i_{(k)} L^j_{(k)}.$$

Now, using the relation (8), we get

$$A_{ij} L^i_{(k)} L^j_{(k)} = \lambda_{(k)} e_{(k)}.$$

Similarly, from (16), we have

$$A_{ij}L^i_{(k)}L^j_{(m)} = \lambda_{(k)}g_{ij}L^i_{(k)}L^j_{(m)}$$

and now using the relation (16), we get the required result, viz,

$$A_{ij}L^i_{(k)}L^j_{(m)} = 0.$$ **Proved**

36.31 PERMUTATION SYMBOLS AND TENSORS

The **permutation symbol** is written as e_{ijk} and in the Euclidean three dimensional space V_3 is defined by

$$e_{ijk} = \begin{cases} 0\ , & \text{if any two of } i,\ j,\ k \text{ are equal} \\ 1, & \text{If } i,\ j,\ k \text{ is a cyclic permutation} \\ -1\ , & \text{if } i,\ j,\ k \text{ is anti-cyclic permutation} \end{cases} \quad \text{...(1)}$$

Thus,

$$\begin{aligned} &e_{112} = e_{113} = e_{221} = e_{223} = e_{331} = e_{332} = e_{111} = e_{222} = e_{333} = 0 \\ &e_{123} = e_{231} = e_{312} = 1 \\ &e_{132} = e_{321} = e_{213} = -1 \end{aligned} \quad \text{...(2)}$$

We now introduce the entities defined by

$$\varepsilon_{ijk} = \sqrt{g}\,e_{ijk};\ \varepsilon^{ijk} = \frac{1}{\sqrt{g}}e_{ijk}, \quad \text{...(3)}$$

where g is the determinant of the metric tensor g_{ij} of the space referred, which may not necessarily be rectangular. We shall now prove that although e_{ijk} is not a tensor, in general, both ε_{ijk} and ε^{ijk} are tensors, covariant and contravariant respectively, and are called **permutation tensors in three dimensional** space. The generalization to higher dimensions is possible. It is clear from the definitions of e_{ijk}, ε_{ijk} and ε^{ijk} that they are skew-symmetric in all their indices.

Theorem 4. The entities defined by (permutation tensors)

$$\varepsilon_{ijk} = \sqrt{g}\,e_{ijk},\ \varepsilon^{ijk} = \frac{1}{\sqrt{g}}e_{ijk},$$

are respectively covariant and contravariant tensors, where e_{ijk} is a permutation symbol and g is the determinant of the metric tensor g_{ij}.

Proof : We see that

$$e_{ijk}\frac{\partial x^i}{\partial \bar{x}^1}\frac{\partial x^j}{\partial \bar{x}^m}\frac{\partial x^k}{\partial \bar{x}^n} = e_{jik}\frac{\partial x^j}{\partial \bar{x}^1}\frac{\partial x^i}{\partial \bar{x}^m}\frac{\partial x^k}{\partial \bar{x}^n}$$

(interchanging the dummy indices i and j)

$$= -e_{ijk}\frac{\partial x^i}{\partial \bar{x}^m}\frac{\partial x^j}{\partial \bar{x}^l}\frac{\partial x^k}{\partial \bar{x}^n}$$

(using the skew-symmetric property of e_{ijk})

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This show that $e_{ijk} \frac{\partial x^i}{\partial \overline{x}^l} \frac{\partial x^j}{\partial \overline{x}^m} \frac{\partial x^k}{\partial \overline{x}^n}$ is skew-symmetric in l and m,

Similarly, it can be shown that it is skew-symmetric in all l, m and n. But this expression, apart from sign, is the Jacobian determinant $\left|\frac{\partial x^r}{\partial \overline{x}^s}\right|$. From the theory of determinants, it therefore follows that

$$e_{ijk} \frac{\partial x^i}{\partial \overline{x}^l} \frac{\partial x^j}{\partial \overline{x}^m} \frac{\partial x^k}{\partial \overline{x}^n} = e_{lmn} \left|\frac{\partial x^r}{\partial \overline{x}^s}\right| \qquad ...(4)$$

Now, by covariant law we know that

$$\overline{g}_{pq} = \frac{\partial x^i}{\partial \overline{x}^p} \frac{\partial x^j}{\partial \overline{x}^q} g_{ij}.$$

Therefore,

$$\left|\overline{g}_{pq}\right| = \left|\frac{\partial x^i}{\partial \overline{x}^p}\right| \left|\frac{\partial x^j}{\partial \overline{x}^q}\right| \left|g_{ij}\right|$$

or

$$\overline{g} = \left|\frac{\partial x^r}{\partial \overline{x}^s}\right|^2 g. \qquad ...(5)$$

Let, in the coordinate system x^{-i}, the entity ε_{ijk} be denoted by $\overline{\varepsilon}_{lmn}$ where

$$\overline{\varepsilon}_{lmn} = \sqrt{\overline{g}}\, e_{lmn} \qquad ...(6)$$

Now, using (2.017) and (2.108), from (2.109) we find

$$\overline{\varepsilon}_{lmn} = \sqrt{g}\ \mathrm{e_{ijk}} \frac{\partial x^i}{\partial \overline{x}^l} \frac{\partial x^j}{\partial \overline{x}^m} \frac{\partial x^k}{\partial \overline{x}^n}$$

$$= \varepsilon_{ijk} \frac{\partial x^i}{\partial \overline{x}^l} \frac{\partial x^j}{\partial \overline{x}^m} \frac{\partial x^k}{\partial \overline{x}^n} \qquad ...(7)$$

This shows that ε_{ijk} is a third order covariant tensor.

Also, writing e^{lmn} for e_{lmn} and e^{-ijk} for e_{ijk} we have

$$\varepsilon^{lmn} = \frac{1}{\sqrt{g}} e^{lmn} = \left|\frac{\partial x^r}{\partial \overline{x}^s}\right| \frac{e^{lmn}}{\sqrt{\overline{g}}} \qquad \text{[using (5)]}$$

$$= \frac{1}{\sqrt{\overline{g}}} e^{-ijk} \frac{\partial x^l}{\partial \overline{x}^i} \frac{\partial x^m}{\partial \overline{x}^j} \frac{\partial x^n}{\partial \overline{x}^k} \qquad \text{[using (4)]}$$

$$= \varepsilon^{-ijk} \frac{\partial x^l}{\partial \overline{x}^i} \frac{\partial x^m}{\partial \overline{x}^j} \frac{\partial x^n}{\partial \overline{x}^k} \qquad ...(6)$$

This shows that ε^{lmn} is a contravariant tensor of the third order. Hence the proposition.

Remark: *In rectangular cartesian coordinates g = 1, therefore the permutation tensors have components as those of permutation symbols and there is no distinction between contravariant and covariant components, i.e.*

$$\varepsilon^{ijk} = \varepsilon_{ijk} = e_{ijk}.$$

Example. *Prove that*

$$\varepsilon_{ijk} = g_{il}\, g_{jm} g_{kn}\, \varepsilon^{lmn}$$

$$g_{il}\, g_{jm} g_{kn}\, \varepsilon^{lmn} = g_{il}\, g_{jm} g_{kn} \frac{1}{\sqrt{g}} e_{lmn} \qquad \text{[By definition (3)]}$$

$$= \frac{1}{\sqrt{g}} g_{il} g_{jm} g_{kn} e_{lmn} \qquad ...(7)$$

But, $g_{il}\, g_{jm} g_{kn}\, e_{lmn} = g_{i1}\, g_{j2} g_{k3} + g_{i2}\, g_{j3} g_{k1} + g_{i3}\, g_{j1} g_{k2}$

$$- g_{i1}\, g_{j3} g_{k2} - g_{i2}\, g_{j1} g_{k3} - g_{i3}\, g_{j2} g_{k1}$$

$$= \begin{vmatrix} g_{i1} & g_{i2} & g_{i3} \\ g_{j1} & g_{j2} & g_{j3} \\ g_{k1} & g_{k2} & g_{k3} \end{vmatrix}$$

$$= g e_{ijk}. \qquad ...(8)$$

Combining (7) and (8), we get

$$g_{il}\, g_{jm} g_{kn}\, \varepsilon^{lmn} = \sqrt{g} e_{ijk} = \varepsilon_{ijk}. \qquad ...(9)$$

Hence the proposition. **Proved.**

36.32 ALTERNATING TENSOR

Consider an abstract entity of order 3 and dimension 3 such that its components relatively to every system of co-ordinate axes are the same and given by $\in_{ijk}$ where

$$\in_{ijk} = \begin{cases} 0 & \text{if any two of } ijk \text{ are equal} \\ 1 & \text{if } ijk \text{ is a cyclic permutation } 1,2,3 \\ -1 & \text{if } ijk \text{ is an anti cyclic permutation } 1,2,3 \end{cases}$$

Thus for unequal values of the suffixes, we have

$$\in_{123} = \in_{312} = \in_{231} = 1, \quad \in_{132} = \in_{213} = \in_{321} = -1$$

Let OX_1, OX_2, OX_3, $O\overline{X}_1, O\overline{X}_2, O\overline{X}_3$ be two systems of rectangular axes. We want to show that $\in_{ijk}$ is a tensor of order three. Consider, now expression

$$l_{ip}\, l_{jq}\, l_{kr} \in_{ijk}$$

For any given system of values *p, q, r,* the expression (1) consists of a sum of $3^3 = 27$ terms of which 6 only are non-zero, for the other 21 terms corresponds to a case when atleast two of *i*, *j*, *k* are equal. The expression (1) can be written as in the form of determinant

$$= \begin{vmatrix} l_{1p} & l_{2p} & l_{3p} \\ l_{1q} & l_{2q} & l_{3q} \\ l_{1r} & l_{2r} & l_{3r} \end{vmatrix}$$

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From properties of determinants,

$$\text{Above determinant} = \begin{cases} 0 & \text{if any two of } p, q, r \text{ have equal value.} \\ 1 & \text{if } p, q, r \text{ is a cyclic permutation of 1, 2, 3} \\ -1 & \text{if } p, q, r \text{ is a non cyclic permutation of 1, 2, 3} \end{cases}$$

Thus we see that the components of the given entity in any two systems of rectangular axes satisfy the tensorial transformation equations so that the entity is a tensor. This tensor is known as *Alternate tensor*. Thus, we see alternate tensor is same as skew-symmetric tensor. $\in_{ijk}$, always denote the alternating tensor.

EXERCISE 36.2

1. Find g and gij corresponding to the metric

$$ds^2 = 5\left(dx^1\right)^2 + 3\left(dx^2\right)^2 + 4\left(dx^3\right)^2 - 6dx^1dx^2 + 4dx^2dx^3.$$

2. Find the values of g and g^{ij}, if

$$ds^2 = \frac{dr^2}{1-\frac{r^2}{R^2}} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$ where R is constant

3. Prove that for an orthogonal co-ordinate system

(*a*) $g_{12} = g_{23} = g_{31} = 0$ (*b*) $g^{11} = \frac{1}{g_{11}}, g^{22} = \frac{1}{g_{22}}, g^{33} = \frac{1}{g_{33}}$

4. Surface of a sphere is a two dimensional Riemannian space. Find its fundamental metric tensor. If a be the fixed radius of the sphere.

5. If the covariant vectors e_i are orthogonal, show that

(*a*) g_{ij} is diagonal, (*b*) $g^{ii} = 1/gii$ (no summation), (*c*) $\left|\varepsilon^i\right| = 1/\left|\varepsilon_i\right|$.

6. Prove that $\left(\varepsilon^i \cdot \varepsilon^j\right)\left(\varepsilon_j \cdot \varepsilon_k\right) = \delta^i_k$.

ANSWERS

1. $g = 4, g^{11} = 2, g^{22} = 5, g^{33} = 5, g^{12} = 3, g^{23} = -2.5, g^{13} = -5$

2. $g = \frac{r^4\sin^2\theta}{1-\frac{r^2}{R^2}}; g^{11} = 1 - \frac{r^2}{R^2}, g^{22} = \frac{1}{r^2}, g^{33} = \frac{1}{r^2\sin^2\theta}, g^{ij} = 0\left(i \neq j\right)$

4. $g_{11} = a^2, g_{22} = a^2\sin^2\theta, g = a^4\sin^2\theta, g^{11} = \frac{1}{a^2}, g^{22} = \frac{1}{a^2\sin^2\theta}, g^{12} = 0 = g^{21}.$

37

CHAPTER

Special Theory of Relativity

37.1 SPECIAL THEORY OF RELATIVITY

Michelson Morley experiment and sits out come;

The Michelson Morley experiment was supposed to be one of the most famous null experiment to detect the presence of other. The experiment bad helped Lorentz, Fitzgerald, Poincare to but their observations and also helped Einstein to describe. The propagation of light thought space and time.

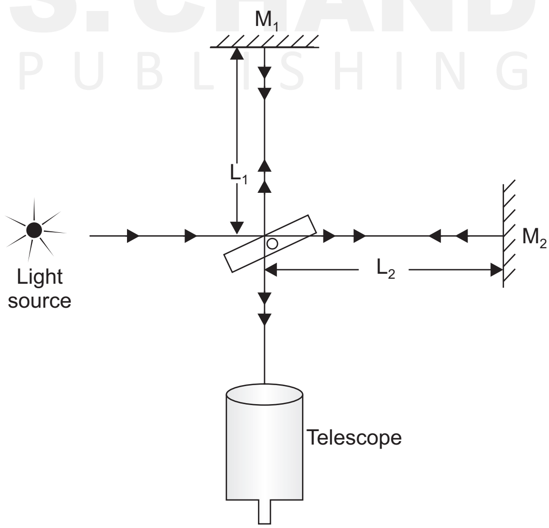
Earlier it was proved that sound needs a medium (water, air, etc) to travel from one wave to another and in 1864 James clerk Maxwell proved that light is an electromagnetic wave and thus it was assumed. that there must exist an other which helps in propagation of light wave. It we assumed that other exists everywhere and is on effected by the Matter. The Michelson - Morley, was conducted in the year 1887 to select the of existence of ether.

The experiment was con wet with the help of two mirrors M, sm. The beam splitter and with a light source and a telescope to observe the interference pattern.

Basis of the experiment:

The experiment was conducted in two stages:

(1) Which the mirrors are at rest position



Let us taken, $L_1 = L_2 = L$

The time taken by light beam for both arms is $\boxed{t = \frac{2}{c}}$

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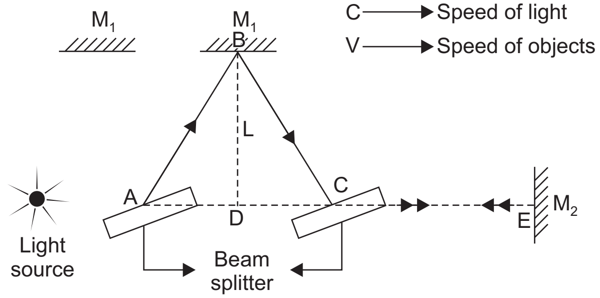
(ii) Where the objects (Mirrors) started moving with a velocity v w. r. t other them



Let us assume that

Time taken by ray from A to B = t_1^1

Time taken by ray from A to B

Then from B to C = t_1

$$\Rightarrow \quad \boxed{t_1^1 = \frac{t_1}{2}}$$

Also AB = ct_1^1

AB = ct_1^1

AD = vt_1^1

In $r + \angle d$ ΔABD

$$AB^2 = AD^2 + BD^2$$

$$\Rightarrow \quad C^2 \; t_1^1 = V^2 \; {t_1^1}^2 + L^2$$

$$\Rightarrow \quad L^2 = (C^2 - V^2) \; {t_1^1}^2$$

$$\Rightarrow \quad L^2 = {t_1^1}^2 \; (C^2 - V^2)$$

Also $\quad t_1^1 = \dfrac{L}{\sqrt{C^2 - V^2}}$

Also $\quad t^1 = 2 \; t_1^1$

$$\Rightarrow \quad t_1^1 = \frac{2L}{\sqrt{C^2 - V^2}}$$

$$\Rightarrow \quad \boxed{t_1 = \frac{2L}{C\sqrt{1 - \frac{V^2}{C^2}}}}$$

Now we calculate the time taken by ray w. r. t M_2

The time taken is

$$t_2 = \frac{\text{CE}}{\text{Relative speed}} + \frac{\text{EC}}{\text{Relative speed}}$$

$$t_1 = \frac{L}{C+V} + \frac{L}{C-V}$$

$$\Rightarrow t_2 = \frac{2CL}{C^2 - V^2}$$

$$\Rightarrow t_2 = \frac{2L}{C}, \frac{C^2}{C^2 - V^2}$$

$$\Rightarrow t_2 = \frac{2L}{C}, \frac{1}{\left(1 - \frac{V^2}{C^2}\right)}$$

The time difference will be

$$\Delta t = t_1 - t_2$$

$$= \frac{2}{C\sqrt{1 - \frac{V^1}{C^1}}} - \frac{2}{C\left(1 - \frac{V^1}{C^1}\right)}$$

$$t = \frac{2L}{C}\left[\frac{1}{\sqrt{1 - \frac{V^2}{C^2}}} - \frac{1}{\left(1 - \frac{V^2}{C^2}\right)}\right]$$

On rotating the positive of mirrors by 90° we get.

$$t^1 = \frac{2L}{C}\left[\frac{1}{\sqrt{1 - \frac{V^2}{C^2}}} - \frac{1}{\left(1 - \frac{V^2}{C^2}\right)}\right]$$

Now the time difference is

$$\Delta t^1 - \Delta t = \frac{2L}{C}\left[\frac{1}{1 - \frac{V^2}{C^2}} - \frac{1}{\sqrt{1 - \frac{V^2}{C^2}}}\right]$$

$$- \frac{2L}{C}\left[\frac{1}{\sqrt{1 - \frac{V^1}{C^1}}} - \frac{1}{\left(1 - \frac{V^2}{C^2}\right)}\right]$$

$$= \frac{2L}{C}\left[\left(1 - \frac{V^2}{C^2}\right)^{-1} - \left(1 - \frac{V^2}{C^2}\right)^{-1/2}\right]$$

$$-\frac{2L}{C}\left[\left(1-\frac{V^2}{C^2}\right)^{-1/2}-\left(1-\frac{V^2}{C^2}\right)^{-1}\right]$$

Using binomial expansion and negleeting the higher times we get

$$t =^1 \Delta t = \frac{2L}{C}\left[1+\frac{V^2}{C^2}-1+\frac{V^2}{2C^2}-+\frac{V^2}{2C^2}+1-\frac{V^2}{C^2}\right]$$

$$t =^1 -\Delta t\left(\frac{2L}{C}\right).\frac{V^2}{C^2}$$

$$T = \left(\frac{2L}{C}\right)\frac{V^2}{C^2}$$

We know that

$$\text{Frequency} = \frac{C}{\lambda}$$

$$\Rightarrow C = \lambda\gamma$$

$$\Rightarrow C = \frac{\lambda}{T}$$

$$\Rightarrow T = \frac{\lambda}{C}$$

$$\Rightarrow \frac{2L}{C}.\frac{C^2}{V^2} = \frac{\lambda}{C}$$

$$\boxed{\Rightarrow \frac{\lambda}{\alpha} = L.\left(\frac{C}{V}\right)^2}$$

Where $\frac{\lambda}{\gamma}$ = path difference.

⇒ Interference bringe will shift after a path difference of λ/α .

The value of L was taken as 11m

$\lambda = 5.5\times10^{-7}$m

$\frac{V}{C} = 10^{-4}$

$$\Rightarrow L.\left(\frac{C}{V}\right)^2.\frac{\alpha}{\lambda} = 0.4$$

⇒ It shows 40% shift in interference bringe but actually no shift food hence the theory of existence of ether was nullified.

37.2 FUNDAMENTAL POSTULATE OF EINSTEIN THEORY OF RELATIVITY

(1) The fundamental laws of physics are of the same form in all inertial frames of reference.

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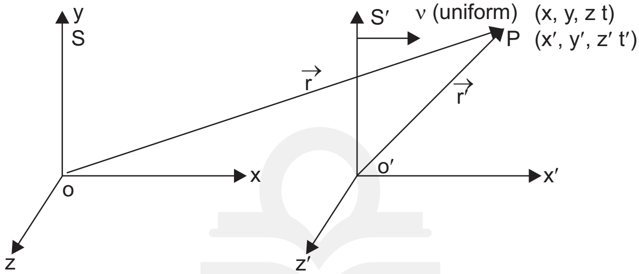
(2) The speed of the light is same in all inertial frames of reference, regardless of the motion of the source relative to the observer.

for Einstein in all frames (space-time) co-ordinates are Relative or changable.

37.3 LORENTZ'S TRANSFORMATION

"These are the equations which enable us to find the relation between the space and the time co-ordinates of an event in two different inertial-frames in uniform relative motion w.r.t each other, in accordance with the Postulate of special Theory of Relativity."



Both are inertial frame.

Suppose $t = t' = 0$ at O, O', when they coincide then a flash of light is sent out from 'O' along x-axis in wave-front. The light wave will Travel outward in all direction with speed 'C' and hence will be an expanding sphere, At any time 't', to the observer of frame 'S', at any time t, the light wave will appear a Sphere of Radius 'ct'', ($c \rightarrow$ in all frame is constant).

Whose equation $t = \dfrac{OP}{C} = \dfrac{\left(x^2+y^2+z^2\right)^{1/2}}{C}$

$$\Rightarrow \quad x^2 + y^2 + z^2 = c^2t^2 \qquad ...(1)$$

and $$t' = \frac{O'P}{C} = \frac{\left(x'^2+y'^2+z'^2\right)^{1/2}}{C}$$

$$\Rightarrow \quad x'^2 + y'^2 + z'^2 = c^2t'^2 \qquad (2)$$

According to Gallilean Transformation

$$x' = x - vt, \quad y' = y, \quad z' = z, \; t' = t$$

So from equation (2) $(x - vt)^2 + y^2 + z^2 = c^2t^2$

$$\Rightarrow \quad x^2 - 2xvt + v^2t^2 + y^2 + z^2 = c^2t^2 \qquad ...(3)$$

Here equation (3) not resembles with (1) Here $(-2xvt + v^2t^2)$ is extra factor. So Gallilean Transformation are not satisfied. Now from the Property of Homogenity and Isotropy of the free space.

(1) Transformations should be linear and simple

(2) at $v << c$ these Transformation change in Gallilean Transformation

$$x' = k\left(x - vt\right) \qquad ...(4)$$

from (1)st Postulate all laws in nature same so

$$x = k(x' + vt') \quad ...(5) \text{ and } y = y' \text{ and } z = z' \qquad ...(6)$$

from (4) & (5) $\quad x = k(x' + vt')$

$$x = k[k(x - vt) + vt']$$

$$x = k^2(x - vt) + kvt' \Rightarrow kvt' = k^2vt + (1 - k^2)x$$

$$\Rightarrow \quad t' = kt + \frac{(1-k^2)}{k}\frac{x}{v} \text{ or } t' = k\left[t + \left(\frac{1}{k^2} - 1\right)\frac{x}{v}\right] \qquad ...(7)$$

from (2)nd Postulate C is same in all frames so

$$x = ct, \; x' = ct' \qquad ...(8)$$

from (4), (7), (8) so

$$k(x - vt) = ck\left[t + \frac{x}{v}\left(\frac{1}{k^2} - 1\right)\right]$$

or $\cancel{k}x\left[1 - \frac{c}{v}\left(\frac{1}{k^2} - 1\right)\right] = ckt + kvt = \cancel{k}t(c + v)$

$$\cancel{c}t\left[\cancel{1} - \frac{c}{v}\left(\frac{1}{k^2} - 1\right)\right] = \cancel{c}t\left(\cancel{1} + \frac{v}{c}\right)$$

$$\frac{c}{v}\left(1 - \frac{1}{k^2}\right) = \frac{v}{c} \Rightarrow \frac{1}{k^2} = 1 - \frac{v^2}{c^2}$$

$\Rightarrow \quad k = \dfrac{1}{\sqrt{1 - v^2/c^2}}$ is never negative because $v << c$.

from (7) $t' = k\left[t + \left(\cancel{1} - \frac{v^2}{c^2} - \cancel{1}\right)\frac{x}{v}\right] = k\left[t - \frac{vx}{c^2}\right]$

$$\Rightarrow \quad \boxed{t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}}$$

So Lorentz Transformation equations are

Transform S to S' is

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \; y' = y, \quad z' = z, \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - v^2/c^2}}$$

Now take Lorentz Inverse Transformation equations are

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, \; y = y', \quad z = z', \quad t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$$

If $v << c$ so $\frac{v^2}{c^2} << 1$, so they change in to Gallilean Transformation

Numerical: Prove that the spherical wave-front of light is Invarient under Lorentz's Transformation.

or $\quad x^2 + y^2 + z^2 - c^2t^2 = 0.$

$\Rightarrow \quad x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$

$$\left(\frac{x-vt}{\sqrt{1-v^2/c^2}}\right)^2 + y^2 + z^2 - c^2\left[\frac{t-vx/c^2}{\sqrt{1-v^2/c^2}}\right]^2 = 0$$

$$\frac{(x-vt)^2 - c^2\left(t-\frac{vx}{c^2}\right)^2}{1-v^2/c^2} + y^2 + z^2 = 0.$$

$$\Rightarrow \frac{x^2 + v^2t^2 - \cancel{2xvt} - c^2t^2 - \frac{v^2x^2}{c^2} + \cancel{2xvt}}{1-v^2/c^2} + y^2 + z^2 = 0$$

$$\Rightarrow \frac{x^2\left(1-\frac{v^2}{c^2}\right) - c^2t^2\left(1-\frac{v^2}{c^2}\right)}{1-v^2/c^2} + y^2 + z^2 = 0$$

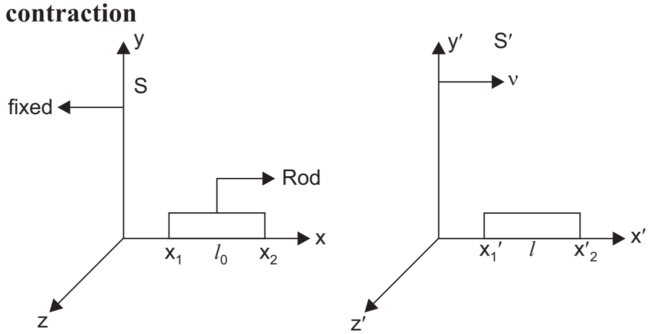
$$\Rightarrow x^2 - c^2t^2 + y^2 + z^2 = 0.$$

Prove that.

38.4 Consequences of Lorentz Transformation

1. Length contraction or Lorentz- Pitzerland Contraction
2. Time dialation
3. Transformation of velocities or Addition of velocities
4. Transformation of Acceleration
5. Relativity of Simultanity
6. Relativity of mass
7. Mass – Energy Equivalance.

1. Length contraction



Consider a Rod lying at rest along x-axis of S frame, so

Proper length of Rod $l_0 = x_2 - x_1$

Now we see measurement in S′ frame which are going in +ve direction from S with velocity 'V' →

So observe length $l = x_2' - x_1'$

Here $x_1 = \dfrac{x_1' + vt'}{\sqrt{1-\dfrac{v^2}{c^2}}}$, $x_2 = \dfrac{x_2' + vt'}{\sqrt{1-\dfrac{v^2}{c^2}}}$

So $x_2 - x_1 = \dfrac{x_2' - x_1'}{\sqrt{1-v^2/c^2}} \Rightarrow \boxed{l_0 = \gamma l}$ $l_0 > l$

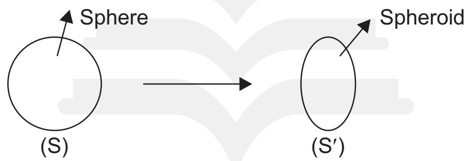
$$\left[\therefore \gamma = \frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}\right]$$

or $\gamma > 1$

But in perpendicular directions $y' = y$, $z' = z$ no change in length

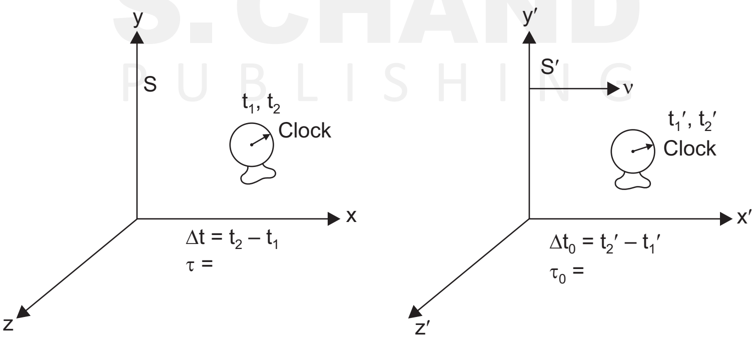
Thus length of the Rod in all other frame of Reference in uniform motion with respect to the frame in which the Rod is at Rest, is Shorter than its Proper length.

Example. A Sphere will look like as a spheroid due to decrease in its diameter Parallel to x-axis.



2. Time-dilation

(Relativity of time)



Consider a clock placed at the O paint *x'* in the frame S′ moving with uniform velocity '*v*' along x-axis with respect to frame 'S′. Suppose at any instant, observer of frame S' for which clock is at rest, have time t_1' So the observer of frame 'S' will find the time to be

$$\boxed{t_1 = \frac{t_1' + \dfrac{vx'}{c^2}}{\sqrt{1-\dfrac{v^2}{c^2}}}}$$

If some time later, the observer of frame S' notes the time as t_2' the observer of frame S will record it as $\boxed{t_2 = \dfrac{t_2' + \dfrac{vx'}{c^2}}{\sqrt{1-\dfrac{v^2}{c^2}}}}$

So $t_2 - t_1 = \dfrac{t_2' - t_1'}{\sqrt{1-\dfrac{v^2}{c^2}}} \Rightarrow \tau = \dfrac{\tau_0}{\sqrt{1-\dfrac{v^2}{c^2}}} = y\tau_0$ and $\tau > \tau_0$

So time interval noted by an observer w.r.t. whom the clock is at rest is smaller than the time interval noted by the observer w.r.t whom clock is in motion.

4. velocity Transformation (Relativistic Addition of velocities) →

Consider a body moving with a constant linear velocity u w.r.t. S frame along *x*-axis and u' w.r.t S' frame along x' – axis. Frame S' moves with velocity V in same direction w.r.t frame *S*.

Suppose U_x, U_y, U_z are component of velocity w.r.t frame *S* and u_x', u_y', u_z' are component of velocity w.r.t frame *S'*.

Here *S* frame fixed.

In *S* frame $u_x = \dfrac{dx}{dt},\ u_y = \dfrac{dy}{dt},\ u_z = \dfrac{dz}{dt}$

In *S'* frame $u_x' = \dfrac{dx'}{dt},\quad u_y' = \dfrac{dy'}{dt},\quad u_z' = \dfrac{dz'}{dt}$

According to Lorentz Transformation

$$x' = \frac{x - vt}{\sqrt{1-\dfrac{v^2}{c^2}}},\ y' = y,\ z' = z,\quad t' = \frac{t - vx/c^2}{\sqrt{1-v^2/c^2}}$$

Now differentiate it

$$dx' = \frac{dx - vdt}{\sqrt{1-v^2|c^2}},\ dy' = dy,\ dz' = dz,\quad dt' = \frac{dt - \dfrac{vdx}{c^2}}{\sqrt{1-v^2/c^2}}$$

Now $$u_x' = \frac{dx'}{dt'} = \frac{dx - vdt}{dt - \dfrac{vdx}{c^2}} = \frac{\dfrac{dx}{dt} - v}{1 - \dfrac{v}{c^2}\dfrac{dx}{dt}} = \frac{u_x - v}{1 - \dfrac{v}{c^2}u_x} \quad ...(1)$$

$$u_y' = \frac{dy'}{dt'} = \frac{dy\sqrt{1-v^2/c^2}}{dt - \dfrac{vdx}{c^2}} = \frac{dy/dt\sqrt{1-v^2/c^2}}{1 - \dfrac{v}{c^2}\dfrac{dx}{dt}}$$

$$u_y' = \frac{u_y\sqrt{1-v^2/c^2}}{1 - \dfrac{v}{c^2}u_x} \quad ...(2)$$

Similarly $u_z' = \dfrac{u_z\sqrt{1-v^2/c^2}}{1-\dfrac{v}{c^2}u_x}$ ...(3)

So equations (1) (2) and (3) give Transformation equations for velocity components in S to S' frame.

The inverse velocity Transformation equations from S′ to S frame is

$$u_x = \frac{u_x' + v}{1+\dfrac{vu_x'}{c^2}}, \quad u_y = \frac{u_y'\sqrt{1-v^2/c^2}}{1+\dfrac{v}{c^2}u_x'}, \quad u_z = \frac{u_z'\sqrt{1-v^2/c^2}}{1+\dfrac{v}{c^2}u_x'}$$

Case 1. when $v << c$ so equation (1) (2) and (3) are

$$u_x' = u_x - v, \quad u_y' = u_y, \quad u_z' = u_z.$$

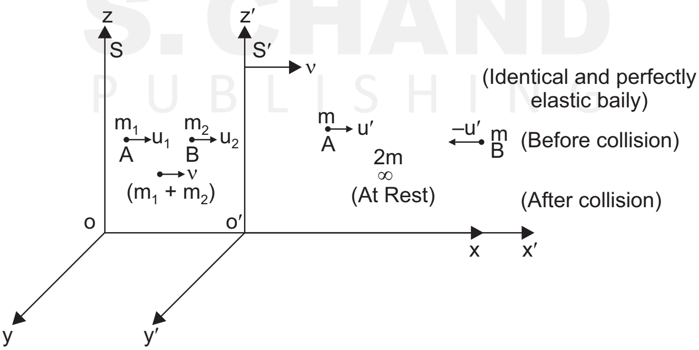
Called classical (Newtonian) Gallilean Law of addition of velocity.

Case 2. If we consider the particle to be a photon moving with velocity ‘C’ in fame S’ which is also moving with velocity ‘C′ along x-axis so $u_x = \dfrac{u_x' + v}{1+\dfrac{vu_x'}{c^2}}$

If $u_x' = c \quad u_x = \dfrac{c+v}{1+\dfrac{vc}{c^2}} = \dfrac{c+v}{1+\dfrac{v}{c}} = c$

So speed of light is an absolute constant, independent of the motion of the frame of reference and all frame of Reference.

5. Relativity of mass (variation of mass with velocity)



According to newtonic (classical) mechanics mass of moving Particle does not depend on velocity. But Relatively see below.

Suppose to the observer of frame S, the masses of the bodies ‘A’ and ‘B’ appears to be m_1 and m_2 and velocities u_1 and u_2 after collision the two bodies come to rest momentarily in frame S' they together will appear to be moving with the velocity of frame S' with velocity v to the observer of frame S.

According to law of conservation of momentum is

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v$$

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$$\frac{m_1}{m_2} = \left(\frac{v - u_2}{u_1 - v}\right) \qquad \text{...(1)}$$

from inverse velocity Transformation equation we have

$$u_x = \frac{u'_x + v}{1 + \frac{vu_x'}{c^2}}$$

The Body A moves with velocity u' in frame S' and appears to be moving with velocity u_1 to the observer of frame S.

Set $u'_x = u'$ and $u_x = u_1$

So velocity of Body A in frame S is $u_1 = \dfrac{u' + v}{1 + \frac{vu'}{c^2}}$ ...(2)

Similarly set $u'_x = -u'$ and $u_x = u_2$

So velocity of Body 'B' in frame S is $u_2 = \dfrac{-u' + v}{1 - \frac{vu'}{c^2}}$ ...(3)

From equation (2) & (3) in (1)

$$\frac{m_1}{m_2} = \frac{v - \left(\dfrac{-u' + v}{1 - \frac{vu'}{c^2}}\right)}{\dfrac{u' + v}{1 + \frac{vu'}{c^2}} - v} = \frac{1 + \frac{vu'}{c^2}}{1 - \frac{vu'}{c^2}} \qquad \text{...(4)}$$

Now from equation (2)

(Tricky Point)

$$1 - \frac{u_1^2}{c^2} = 1 - \frac{1}{c^2}\left[\frac{u' + v}{1 + \frac{vu'}{c^2}}\right]^2 = \frac{\left(1 + \frac{vu'}{c^2}\right)^2 - \frac{1}{c^2}(u' + v)^2}{\left(1 + \frac{vu'}{c^2}\right)^2}$$

$$\left(1 - \frac{u_1^2}{c^2}\right) = \frac{1 + \frac{v^2 u'^2}{c^4} - \frac{u'^2}{c^2} - \frac{v^2}{c^2}}{\left(1 + \frac{vu'}{c^2}\right)^2} = \frac{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{u'^2}{c^2}\right)}{\left(1 + \frac{vu'}{c^2}\right)^2}$$

$$\Rightarrow \left(1 + \frac{vu'}{c^2}\right)^2 = \frac{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{u'^2}{c^2}\right)}{\left(1 - \frac{u_1^2}{c^2}\right)}$$

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$$\Rightarrow \quad \left(1+\frac{vu'}{c^2}\right)=\sqrt{\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{\left(1-\frac{u_1^2}{c^2}\right)}} \quad ...(5)$$

Similarly from equation (5) $u' \to -u' \quad \& \left(u_1 \leftrightarrow u_2\right)$

$$\left(1-\frac{vu'}{c^2}\right)=\sqrt{\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{\left(1-\frac{u_2^2}{c^2}\right)}} \quad ...(6)$$

Equation (4) & (5) Put in (4)

$$\frac{m_1}{m_2}=\frac{\sqrt{1-u_2^2/c^2}}{\sqrt{1-u_1^2/c^2}}$$

Now If w.r.t. S frame, Before collision velocity of Particle B is zero

So $u_2 = 0$.

$$\frac{m_1}{m_2}=\frac{1}{\sqrt{1-u_1^2/c^2}} \quad \text{or} \quad m_1=\frac{m_2}{\sqrt{1-\frac{u_1^2}{c^2}}}$$

So Body 'B' at Rest so $m_2 = m_0$

$$\boxed{m_1=\frac{m_0}{\sqrt{1-\frac{u_1^2}{c^2}}}}$$

Hence above equation can be considered to be applicable ot a single body whose rest mass is no and moves with velocity v so $m=\frac{m_0}{\sqrt{a-\frac{v^2}{c^2}}}$

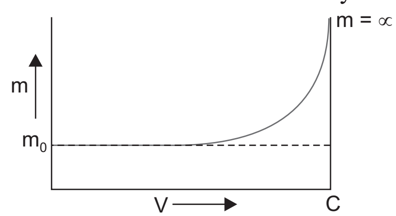
Discussion of Result

(1) When $v << c$ so $m = m_0$ (likely classical Mechanics)

(2) If $v = c$ so $m = \alpha$ imaginary (which is impossible)

If $v > c$ so m = imaginary (which is impossible)

also say when velocity incase so effective mass of body increased.



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Experimental verification

(1) For high energy electrons and β-particles emitted by same radio –active substance by bunchier, Kauffmann, guge and lavanchy)

(2) Splitting of spectral line in μ-spectrum and phenomenon of fine – structure of H- spectrum by Summerfield Relativistic Co-reaction.

(3) Particle accelerator (cyclotron, betation) have mass increase with velocity increase.

Mass–Energy Equivalence

Mass is depend on velocity so K.E is also change with velocity use Newton second law & work energy the omen both are invariant in all frame by 1st postulate.

Suppose a force 'F' act over a body whose Rest mass is 'm_0' over a distance dx, the amount of work done by the force will appear as increase in K.E (dt)

So $$dt = Fdx \quad ...(1)$$

We know $f = \frac{d\vec{p}}{dt} = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt}$

Here m & v variable and m_0 & c are constant Quantity.

So $dT = m\frac{dv}{du}dx + v\frac{dm}{dt}dx$

$dT = mv\ dv + v^2 dm$ ...(2) $\quad$ ($\because$ $v = \frac{dx}{dt}$ at any instant)

we know $m = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}$ or $\frac{{m_0}^2}{1-\frac{v^2}{c^2}}$

$$\Rightarrow m^2 = \frac{{m_0}^2 c^2}{c^2 - v^2} \Rightarrow m^2c^2 - m^2v^2 = {m_0}^2 c^2 \quad ...(3)$$

Differentials equation (3) $2mc^2 dm - \left(2mv^2 dm + m^2 2vdv\right) = 0$

$$C^2 dm - (mvdv + v^2 dm) = 0$$

$$\Rightarrow \quad mvdv + v^2 dm = c^2 dm \quad ...(4)$$

From equation (2) & (+) $\boxed{dT = c^2 dm}$ ...(5)

It shows that change in ($K.\ E$) of a body can be expressed in terms of change in its mass due to motion.

When a body is accelerated from rest to a velocity 'V' it's mass increases from m_0 t_0 m and K.M acquired is obtained by integrating equation (5) between the limts m_0 to m Therefore

$$T = \int_{m_0}^{m} c^2 dm = c^2(m - m_0)$$

So K.E energy of moving particle is equal to c^2 times the gain in mass due to motion.

m_0 is Rest mass of the particle and $m_0 c^2$ is Rest energy called internal energy.

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So total energy $E = T + m_0c^2 = C^2\,(m - m_0) + m_0C^2 = m\,c^2$ ...(6)

So $E = mc^2$ is Einstein mass – Energy equivalence theorem.

Discussion of the Result

(1) Relation $E = mc^2$ shows that equivalence of mass and energy so thus special theory of Relativity ascribes energies to all masses and masses to all energies.

(2) In classical mechanics, the law of conservation of mass and energy are two separate peinciple independent of each other The Relations E = mc^2 leads to unified ion of the two laws into one law called low of conservation of Relativistic energy.

(3) In classical mechanics mass is considered something fundamental to matter while energy is a property of the matter acquired by virtue of its position or motion. The Relation $E = mc^2$ puts an end to such a distinction between mass and energy.

(4) The kinetic energy of a particle travelling with a velocity v is

$$T = c^2\,(m - m_0)$$

Here $m = \dfrac{m_0}{\sqrt{1-v^2/c^2}}$

So $$T = c^2\left[m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2} - m_0\right]$$

$$T = m_0c^2\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right]$$

$$T = m_0c^2\left[\left(1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}+---\right)-1\right] \qquad [\because](1-x)^{-n} = 1+nx+\frac{n(n+1)}{2}\times 2$$

$$T = \frac{1}{2}m_0v^2 + \frac{3}{8}m_o\frac{v^4}{c^2} + ---$$

When $v << c$

So $T \simeq \frac{1}{2}m_ov^2$ → formula of $K.E$ is classical picture.

Experimental evidence in support of the Mass–Energy Equivalance

(1) For electron $m_0 = 9.1 \times 10^{-31}$ kg

So $E = mc^2 = \dfrac{9.1\times10^{-31}\times9\times10^{16}}{1.6\times10^{-13}} Mev = .511 Mev$ [∴ lev = $1.6 \times 10^{-19} j$]

For 1 amu = 1.67×10^{-27}kg

So **1 amu = 931 Mev**

(2) Pair – production and An illation of matter also support the equivalence of mass and energy.

(3) Fission and fusion processes are the direct applications of the Einstein's mass – energy Relation.

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EXERCISE 38.1

Chapter-end Exercises

MULTIPLE CHOICE QUESTIONS

1. According to the special theory of relativity, something that happens at a particular point in space at a particular instant of time is called ....

(*i*) Event (*ii*) Phenomenon (*iii*) Incident (*iv*) Happening

2. According to the special theory theory of relativity, a person or equipment meant to observe and take measurement about the event is called ....

(*i*) Supervisor (*ii*) Observer (*iii*) Examiner (*iv*) Invigilator

3. A frame of reference is specified by a

(*i*) Observer (*ii*) Decimal system (*iii*) coordinate system (*iv*) metric system

4. The frame of reference in which the law of inertia is satisfied is called .... frame of reference.

(*i*) Einstein's (*ii*) Newton's (*iii*) Non-inertial (*iv*) Inertial

5. The frame of reference in which the law of inertia is not satisfied is called .... frame of reference.

(*i*) Einstein's (*ii*) Inertial (*iii*) Non-nertial (*iv*) Newton's

6. A car moving with a constant velocity represent .... frame of reference.

(*i*) Einstein's (*ii*) Inertial (*iii*) Non-inertial (*iv*) Newton's

7. According to the special theory of relativity, physical laws are the same in frames of reference which

(*i*) move at uniform velocity (*ii*) accelerate

(*iii*) move in circles (*iv*) move in ellipses.

8. clocks in a moving reference frame, compared to identical clocks in a stationary frame, appear to run

(*i*) Slower (*ii*) At the same rate (*iii*) Faster (*iv*) Backward in time

9. A spaceship, moving away from the Earth at a speed of $0.9c$, fires a light beam backward. An observer on Earth would see the light arriving at a speed of.

(*i*) $0.1c$ (*ii*) More than $0.1c$ but less than c

(*iii*) c (*iv*) More than c but less than $1.9c$

10. The term "relativistic" refers to effects that are.

(*i*) Observed when speeds are near the speed of light.

(*ii*) Noticed about a moving object.

(*iii*) Observed when objects move backward in time.

(*iv*) Measured by stationary observers only.

11. The purpose of the Michelson- Morley experiment was to

(*i*) Determine the velocity of light.

(*ii*) Detect possible motion of the Earth relative to the sun.

(*iii*) Detect possible motion of the sun relative to the ether.

(*iv*) Detect possible motion of the Earth relative to the ether.

12. A spaceship approaches the Moon, traveling at $0.5c$ with respect to the moon. It crew shines a laser at the Moon. The beam strikes a lunar mirror and is reflected back to the ship. The crew on the ship will measure the speed of the reflected Beam to be

(*i*) $2.0c$ (*ii*) $1.5c$ (*iii*) c (*iv*) $0.75c$

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13. A train has a rest length of 100m. the traveling at a very high velocity, it goes through a tunnel of length 80m. observers located at both ends of the tunnel . what is the Velocity of the train expressed in units of *c*?

(*i*) 0.866*c* (*ii*) 0.33*c* (*iii*) 0.50*c* (*iv*) 0.60*c*

14. Relative to its period on the earth, the period a pendulum on the moon is

(*i*) Shorter (*ii*) Longer
(*iii*) The same as on the earth (*iv*) Varies with time

15. Lorentz transformations are converted into Galilean transformation for .... particle.

(*i*) Large mass (*ii*) Small velocity (*iii*) Large velocity (*iv*) Small mass

16. According to the special theory of relativity a moving clock always go

(*i*) Slow (*ii*) Down (*iii*) Up (*iv*) Fast

17. The energy momentum relation in special theory of relativity is given by

(*i*) $E = \sqrt{m_0^2 c^4 + c^2 p^2}$ (*ii*) $E = \sqrt{m_0^4 c^4 + c^4 p^4}$ (*iii*) $E = \sqrt{m_0^2 c^4 - c^2 p^2}$ (*iv*) $E = \sqrt{m_0 c^2 + c^2 p^2}$

18. Calculate the velocity of a body if its total energy is three times its rest energy

(*i*) 0.54*c* (*ii*) 0.76*c* (*iii*) 0.94*c* (*iv*) none of these

19. The relativistic mass expression is given by ....

(*i*) $m' = \frac{m_0}{1 - \frac{u^2}{c^2}}$ (*ii*) $m' = \frac{E_0}{1 - \frac{u^2}{c^2}}$ (*iii*) $m' = \frac{\mu_0}{1 - \frac{u^2}{c^2}}$ (*iv*) $\mu' = \frac{m_0}{1 - \frac{u^2}{c^2}}$

20. Lorentz transformation of momentum for Y component ....

(*i*) P′Y = PY (*ii*) P′Y = Pz (*iii*) P′Y = Ex (*iv*) P′Y = Bx

21. The speed of light is represented by ....

(*i*) E (*ii*) M (*iii*) Q (*iv*) C

22. According to Einstein's special Theory of Relativity, laws of physics can be formulated based on ....

(*i*) Inertial frame of Reference (*ii*) Non inertial frame of Reference
(*iii*) Both non and inertial frame of Reference (*iv*) Quantum state

23. As an object approaches the speed of light, it's mass becomes ....

(*i*) Zero (*ii*) Double (*iii*) Remain same (*iv*) Infinite

24. In relativity an electric field and magnetic fields are ....

(*i*) Dependent (*ii*) Independent (*iii*) Interdependent (*iv*) Null

25. A charged particle in an electromagnetic field experience a force called ....

(*i*) Gravitational forces (*ii*) Lorentz force (*iii*) Frictional force (*iv*) Restoring force

26. The electric force is represented as ....

(*i*) $F = qE$ (*ii*) $F = qE + q(uxB)$ (*iii*) $F = q(E - uB)$ (*iv*) $F = 0$

27. The Maxwell first equation is knows as .... law.

(*i*) Coulombs (*ii*) Newtons (*iii*) Gauss (*iv*) Keplers

28. Which of the following is Einstein's mass energy relation?

(*i*) $E_k = (m - m_0)c^2$ (*ii*) $E^2 - p^2c^2 = m_0{}^2c^4$ (*iii*) $E_k = mv^2/c^2$ (*iv*) $E = mc^2$

29. Relative to a stationary observer, a moving object ....

(*i*) Appears longer than normal
(*ii*) Can do any of the above. It depends on the relative velocity between the observer and the object
(*iii*) Appears shorter than normal.
(*iv*) Keeps its same length time

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30. In the classical mechanics the kinetic expression of a particle of mass *m* and ....

(*i*) Force (*ii*) Moving with velocity *u*

(*iii*) Momentum (*iv*) Acceleration

Answers to Selected Questions

1. (*i*)	**2.** (*ii*)	**3.** (*iii*)	**4.** (*iv*)	**5.** (*iii*)
6. (*ii*)	**7.** (*i*)	**8.** (*i*)	**9.** (*iii*)	**10.** (*i*)
11. (*iv*)	**12.** (*iii*)	**13.** (*iv*)	**14.** (*ii*)	**15.** (*ii*)
16. (*i*)	**17.** (*i*)	**18.** (*iii*)	**19.** (*i*)	**20.** (*iii*)
21. (*iv*)	**22.** (*i*)	**23.** (*iv*)	**24.** (*iii*)	**25.** (*ii*)
26. (*i*)	**27.** (*iii*)	**28.** (*iv*)	**29.** (*iv*)	**30.** (*ii*)

SHORT ANSEWER TYPE QUESTIONS

1. Define Event and Observer
2. Define Inertial frame of reference and Non-Inertial frame of reference.
3. Derive the energy-momentum relationship for a particle moving at relativistic speed.
4. Write a short note on aluminiferous ether.
5. State the two postulates of special theory of relativity.
6. Discuss the major conclusions of Michelson-Morley experiment.
7. Write the equations for Galilean transformation equations.
8. Write the equations for the Lorentz transformation equations.
9. With the help of an example explain why Lorentz Fitzgerald length contraction is not applicable to the objects which are not moving with relativistic speed.

LONG ANSWER TYPE QUESTIONS.

1. Define frame of reference and discuss the inertial an non- inertial frames of references with the help of necessary diagrams.
2. Discuss the Galilean transformation equations in detail
3. Explain the concept of aluminiferous ether and state the postulates of theory of relativity.
4. What is aluminiferous ether? Discuss the Michelson-Morley experiment for the search of either, derive the necessary equations and state its major conclusions.
5. With the help of necessary diagram discuss the Michelson- Morley experiment and enlist its major outcomes.
6. Explain the failure of Galilean transformation equations and derive the Lorentz transformation equation with the help of necessary diagrams and equations.
7. Discuss the phenomenon of Lorentz-Fitzgerald length contraction along with an example.
8. Write a detailed note on Time Dilation.
9. Explain why a moving clock (at a relativistic speed) appears to go slow.
10. Derive the expression for the kinetic energy of a particle moving at relativistic speed and hence establish the relationship showing the equivalence of its mass and energy.
11. Obtain the energy-momentum relationship for a particle moving at relativistic speed.

CHAPTER 38

Calculus of Variation

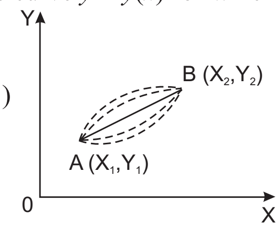
38.1 INTRODUCTION

The calculus of variations primarily deals with finding maximum or minimum value of a definite integral involving a certain function.

38.2 FUNCTIONALS

A simple example of functional is the shortest length of a curve through two points $A(x_1, y_1)$ and $B(x_2, y_2)$. In other words, the determination of the curve $y = y(x)$ for which $y = (x_1) = y_1$, $y\ (x_2) = y_2$ such that

$$\int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{...(1)}$$



is a minimum.

An integral such as (1) is called a *Functional*.

In general, it is required to find the curve $y = y\ (x)$ where $y\ (x_1) = y_1$ and $y\ (x_2) = y_2$ such that for a given function $f\left(x, y, \frac{dy}{dx}\right)$,

$$\int_{x_1}^{x_2} f\left(x, y, \frac{dy}{dx}\right) dx \qquad \text{...(2)}$$

is maximum or minimum.

Integral (2) is known as the functional.

In differential calculus, we find the maximum or minimum value of functions. But the calculus of variations deals with the problems of maxima or minima of functionals.

A functional $I\ [y\ (x)]$ is said to be linear if it satisfies.

(i) $I\ [cy\ (x)] = c\ I\ [y\ (x)]$, where c is an arbitrary constant.

(ii) $I\ [y_1\ (x) + y_2\ (x)] = I\ [y_1\ (x)] + I\ [y_2\ (x)]$, where $y_1\ (x) \in M$ and $y_2\ (x) \in M$

38.3 DEFINITION

A functional $I\ [y\ (x)]$ is maximum on a curve $y = y\ (x)$, if the values of $I\ [y\ (x)]$ on any curve close to $y = y_1\ (x)$ do not exceed $I\ [y_1\ (x)]$. It means $\Delta I = I\ [y\ (x)] - I\ [y_1\ (x)] \le 0$ and $\Delta I = 0$ *on* $y = y_1\ (x)$.

In case of minimum of $I\ [y\ (x)]$, $\Delta I = 0$.

Extremal: A function $y = y\ (x)$ which extremizes a functional is called extremal or extremizing function.

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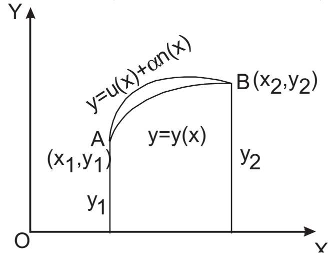
38.4 EULER'S EQUATION IS

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$$

(*D.U. II Sem. 2012*)



This is the necessary condition for

$I=\int_{x_1}^{x_2} f(x,y,y')\,dx$ to be maximum or minimum.

Proof: Let $y = y(x)$ be the curve AB which makes the given function I an extremum.

Consider a family of neighbouring curves

$$Y = u(x) + \alpha\eta(x) \qquad ...(1)$$

where α is a parameter, and $\eta(x)$ is an arbitrary differentiable function.

At the end points A and B,

$$\eta(x_1) = \eta(x_2) = 0$$

when α = 0, neighbouring curves become $y = y(x)$, which is extremal.

The family of neighbouring curves is called the family of *comparison functions.*

If in the functional $\int_{x_1}^{x_2} f(x,y,y')\,dx$ We replace y by Y, we get

$$\int_{x_1}^{x^2} f(x,Y,Y')\,dx=\int_{x_1}^{x_2} f\left[x,y(x)+\alpha\eta(x),y'(x)+\alpha\eta'(x)\right]dx.$$

which is a function of α, say $I(\alpha)$.

$$\therefore \qquad I(\alpha)=\int_{x_1}^{x_2} f(x,Y,Y')\,dx$$

For α = 0, the neighbouring curves become the extremal, an extremum for α = 0.

The necessary condition for this is $I'(\alpha) = 0$ ...(2)

Differentiating I under the integral sign by Leibnitz's rule, we have

$$I'(\alpha)=\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha}+\frac{\partial f}{\partial Y}\frac{\partial Y}{\partial \alpha}+\frac{\partial f}{\partial Y'}\frac{\partial Y'}{\partial \alpha}\right)dx$$

$$I'(\alpha)=\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial Y}\frac{\partial Y}{\partial \alpha}+\frac{\partial f}{\partial Y'}\frac{\partial Y'}{\partial \alpha}\right)dx \qquad \left(\frac{\partial x}{\partial \alpha}=0 \text{ as } \alpha \text{ is indenpendent of } x\right) \qquad ...(3)$$

On differentiating (1), *w.r.t.*'x', we get, $Y'=y'(x)+\alpha\eta'(x)$

Again differentiating *w.r.t.*'α', we get $\frac{\partial Y'}{\partial \alpha}=\eta'(x)$

Differentiating (1), *w.r.t.*, we get $\alpha\frac{\partial Y}{\partial \alpha}=\eta(x)$

Now (3) becomes $I'(\alpha)=\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial Y}\eta(x)+\frac{\partial f}{\partial Y'}\eta'(x)\right]dx$

Integrating the second term on the right by parts, we get

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$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial Y}\eta(x)dx + \left[\left\{\frac{\partial f}{\partial Y'}\eta(x)\right\}_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial Y'}\right)\eta(x)dx\right]$$

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial Y}\eta(x)dx + \left[\frac{\partial f}{\partial Y'}\eta(x_2) - \frac{\partial f}{\partial Y'}\eta(x_1)\right] - \int_{x_1}^{x_2} \frac{d}{dx}\left[\frac{\partial f}{\partial Y'}\right]\eta(x)dx$$

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial Y}\eta(x)dx + 0 - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial Y'}\right)\eta(x)dx = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial Y} - \frac{d}{dx}\left(\frac{\partial f}{\partial Y'}\right)\right]\eta(x)dx\left[\eta(x_1) = \eta(x_2) = 0\right]$$

for extremum value, $I'(\alpha) = 0$

$$0 = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial Y} - \frac{d}{dx}\left(\frac{\partial f}{\partial Y'}\right)\right]\eta(x)dx$$

$\eta(x)$ is an arbitrary continuous function.

$\therefore \quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ which is a required Euler's equation.

Note: Other Forms of Euler's equation

1. $\frac{d}{dx} f(x, y, y') = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx}$

or $$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \quad ...(4)$$

But $$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{\partial f}{\partial y'}y'' \quad ...(5)$$

On subtracting (5) from (4), we have

$$\frac{df}{dx} - \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' - y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

$$\frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'}\right] - \frac{\partial f}{\partial x} = y'\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right] = (y')(0) = 0 \qquad [\textit{Euler's equation}]$$

Hence $$\frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'}\right] - \frac{\partial f}{\partial x} = 0 \quad ...(6)$$

Which is an another form of Euler's equation.

2. We know that $\frac{\partial f}{\partial y'}$ is also a function x, y, y' say $f(x, y, y')$.

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial\phi}{\partial x}\frac{dx}{dx} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} + \frac{\partial\phi}{\partial y'}\frac{dy'}{dx} = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}y' + \frac{\partial\phi}{\partial y'}y''$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y'}\right)y' + \frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial y'}\right)y'' = \frac{\partial^2 f}{\partial x\partial y'} + y'\frac{\partial^2 f}{\partial y\partial y'} + y''\frac{\partial^2 f}{\partial y'^2}$$

Putting the value of $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$ in Euler's equation, we get

$$\frac{\partial f}{\partial y}-\frac{\partial^2 f}{\partial x\,\partial y'}-y'\frac{\partial^2 f}{\partial y\,\partial y'}-y''\frac{\partial^2 f}{\partial y'^2}=0 \qquad ...(7)$$

This is the third form of Euler's equation.

38.5 EXTREMAL

Any function which satisfies Euler's equation is known as Extremal. Extremal is obtained by solving the Euler's equation.

Case 1. If f is independent of $x, i.e., \frac{\partial f}{\partial x}=0.$

On substituting the value of $\frac{\partial f}{\partial x}$ in (6), we have $\frac{d}{dx}\left[f-y'\frac{\partial f}{\partial y'}\right]=0$

Integrating, we get $f-y'\frac{\partial f}{\partial y'}$ = constant

Case 2. When f is independent of y, $i.e., \frac{\partial f}{\partial y}=0.$

Putting the value of $\frac{\partial f}{\partial y}$ in Euler's equation, we get

$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$ = 0, Integrating we get $\frac{\partial f}{\partial y'}$ = **constant**

Case 3. If f is an independent of $y', i.e., \frac{\partial f}{\partial y'}=0$. On substituting the value of $\frac{\partial f}{\partial y'}$ in the Euler's equation, z get $\boldsymbol{\frac{\partial f}{\partial y}=0}$

This is the desired solution.

Case 4. If f is independent of x and y,

we have $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ or $\frac{\partial^2 f}{\partial x\partial y'}=0$ and $\frac{\partial^2 f}{\partial y\partial y'}=0$

Putting these value in Euler's equation (7), we have $y''\frac{\partial^2 f}{\partial y'^2}=0$

If $\frac{\partial^2 f}{\partial y'^2}\neq 0$ then $y''=0$ whose solution is $y = ax + b$.

Example 1. *Write the Euler-Lagrange's equation and explain the terms involved.*

(*D.U. II Sem. 2012, April 2010*)

Solution. The Euler Lagrange's equation is $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)-\frac{\partial f}{\partial y}=0$

Example 2. *Prove that if f does not depend on x explicity, then $f-f'\frac{\partial f}{\partial y'}$ = constant.*

(*D.U. II Sem. 2012*)

Solution. The Euler Lagrange's differential equation is

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$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)-\frac{\partial f}{\partial y}=0$$

Multiplying above equation by y' and adding subtracting the expression $y'' \dfrac{\partial f}{\partial y'}$ (where $y'' = \dfrac{\partial y'}{\partial x}$ and $y' = \dfrac{\partial y}{\partial x}$), we get

$$y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)-y'\frac{\partial f}{\partial y}+y''\frac{\partial f}{\partial y'}-y'\frac{\partial f}{\partial y'}=0 \Rightarrow y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+y''\frac{\partial f}{\partial y'}-y''\frac{\partial f}{\partial y'}-y'\frac{\partial f}{\partial y}=0$$

$$\Rightarrow \quad \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right)-y''\frac{\partial f}{\partial y'}-y'\frac{\partial f}{\partial y}-\frac{\partial f}{\partial x}+\frac{\partial f}{\partial x}=0 \qquad \left(\text{adding and subtracting } \frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \quad \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right)-\left[y''\frac{\partial f}{\partial y'}+y'\frac{\partial f}{\partial y}+\frac{\partial f}{\partial x}\right]+\frac{\partial f}{\partial x}=0$$

$$\Rightarrow \quad \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right)-\frac{\partial f}{dx}+\frac{\partial f}{\partial x}=0 \qquad \{\because f=f\,(y,y',x)\}$$

$$\Rightarrow \quad \frac{d}{dx}\left[y'\frac{\partial f}{\partial y'}-f\right]+\frac{\partial f}{\partial x}=0 \qquad ...(1)$$

If f does not depend upon x explicitly, then $\dfrac{\partial f}{\partial x}=0$ and so we must have

$$\frac{d}{dx}\left[y'\frac{\partial f}{\partial y'}-f\right]=0 \Rightarrow y'\frac{\partial f}{\partial y'}-f= \text{ constant}$$

$$\Rightarrow \quad f-y'\frac{\partial f}{\partial y'}= \text{ constant.} \qquad \textbf{Proved.} \qquad ...(2)$$

Example 3. *Test for an extremum the functional*

$$I\left[y(x)\right]=\int_0^1\left(xy+y^2-2y^2y'\right)dx,\ y(0)=1, y(1)=2$$

Solution. Euler's equation

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0 \qquad ...(1)$$

Here $\quad f=xy+y^2-2y^2y'$

$$\frac{\partial f}{\partial y}=x+2y-4yy' \text{ and } \frac{\partial f}{\partial y'}=-2y^2$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=\frac{d}{dx}(-2y^2)=-4yy'$$

Putting these values in (1), we get $x+2y-4yy'-\left(-4yy'\right)=0$

or $\quad x+2y=0$ or $y=-\dfrac{x}{2}$ At $x=0, y=0$; At $x=1, y=-\dfrac{1}{2}$.

This extremal does not satisfy the boundary conditions $y\,(0)=1$, $y\,(1)=2$.
Hence there is no extremal. **Ans.**

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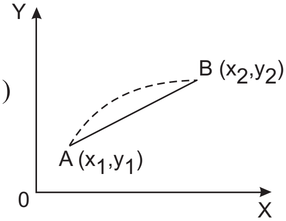
Example 4. *Prove that the shortest distance between two points is along a straight line.*
(*D.U. II Sem. 2012*)

Solution. Let A (x_1, y_1) and B (x_2, y_2) be the two given points and s the length of the arc joining these points.



Then $$s = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_{x_1}^{x_2} \sqrt{(1+y'^2)}\,dx \qquad ...(1)$$

$$y(x_1) = y_1, \quad y(x_2) = y_2$$

If s satisfies the Euler's equation, then it will be minimum

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad \text{(Euler's equation)}$$

Here in (1), $f = \sqrt{(1+y'^2)}$

f is independent of y, i.e., $\dfrac{\partial f}{\partial y} = 0$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{d}{dx}\left(\frac{\partial}{\partial y'}\sqrt{\left(1+y'^2\right)}\right) = \frac{d}{dx}\left[\frac{1}{2}\left(1+y'^2\right)^{-\frac{1}{2}} 2y'\right] = \frac{d}{dx}\frac{y'}{\sqrt{\left(1+y'^2\right)}}$$

Putting these values in Euler's Equation, we have

$$0 - \frac{d}{dx}\frac{y'}{\sqrt{\left(1+y'^2\right)}} = 0 \text{ or } \frac{d}{dx}\frac{y'}{\sqrt{(1+y'^2)}} = 0$$

On integrating $\dfrac{y'}{\sqrt{(1+y'^2)}}$ constant (c), i.e., $(y')^2 = c^2\,(1+y'^2)$

or $$y'^2(1-c^2) = c^2 \text{ or } y'^2 = \frac{c^2}{1-c^2} = m^2 \text{ or } y' = m \text{ or } \frac{dy}{dx} = m$$

Integrating $\quad y = mx + c \qquad ...(2)$

which is a straight line. **Ans.**

Now $\quad y(x_1) = y_1 \quad$ and $\quad y(x_2) = y_2$

$$mx_1 + c = y_1 \text{ and } mx_2 + c = y_2 \qquad ...(3)$$

on subtracting, we get

or $$y_2 - y_1 = m(x_2 - x_1) \text{ or } m = \frac{y_2 - y_1}{x_2 - x_1}$$

Subtracting (3) from (2), we get

$$y - y_1 = m(x - x_1)$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$ **Proved.**

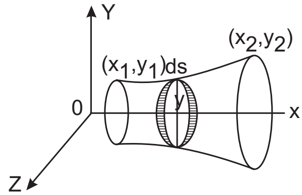
Example 5. *Find the curve connecting the points (x_1, x_2) and (x_2, y_2) which when rotated about the x-axis gives a minimum surface.*
Find the extremal of the functional.

$$\int_{x_1}^{x_2} 2\pi y\, ds \text{ or } 2\pi \int_{x_1}^{x_2} y\sqrt{(1+y'^2)}\, dx$$

Subject to $y(x_1) = y_1,\ y(x_2) = y_2$ **(*D.U. April 2010*)**

Solution. 2π is constant so we have to find the extremal of

$$\int_{x_1}^{x_2} y\sqrt{(1+y'^2)}\, dx$$



Here $f = y\sqrt{(1+y'^2)}$ which is independent of x. $\frac{\partial f}{\partial x} = 0$

One form of Euler's equation is

$$\frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'}\right] - \frac{\partial f}{\partial x} = 0 \quad \text{or} \quad \frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'}\right] = 0 \qquad \left(\frac{\partial f}{\partial x} = 0\right)$$

On integrating, we get, $f - y'\frac{\partial f}{\partial y'} = \text{constant } (c)$ ...(1)

$$f = y\sqrt{(1+y'^2)},\ \frac{\partial f}{\partial y'} = y' = \frac{2y'}{2\sqrt{1+y'^2}}\, y$$

Putting the value of f and $\frac{\partial f}{\partial y'}$ (1), we have

$$y\sqrt{(1+y'^2)} - y'\frac{2y'}{2\sqrt{(1+y'^2)}}\, y = c$$

$$\Rightarrow \quad y\sqrt{(1+y'^2)} - \frac{yy'^2}{\sqrt{(1+y'^2)}} = c \quad \text{or} \quad y(1+y'^2) - yy'^2 = c\sqrt{(1+y'^2)}$$

$$y = c\sqrt{(1+y'^2)} \quad \text{or} \quad y^2 = c^2(1+y'^2)$$

$$\Rightarrow \quad y'^2 = \frac{y^2-c^2}{c^2} \text{ or } y' = \frac{\sqrt{y^2-c^2}}{c} \text{ or } \frac{dy}{dx} = \frac{\sqrt{y^2-c^2}}{c}$$

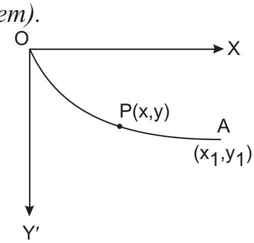
$$\frac{dy}{\sqrt{y^2-c^2}} = \frac{dx}{c} \Rightarrow \int \frac{dy}{\sqrt{y^2-c^2}} = \int \frac{dx}{c} \Rightarrow \cosh^{-1}\frac{y}{c} = \frac{x}{c} + b$$

$y = c\cosh\left(\frac{x}{c} + b\right)$ which is the equation of catenary. This is the required extremal.

Ans.

Example 6. *Find the curve connecting two points (not on a vertical line), such that a particle sliding down this curve under gravity (in absence of resistance) from one point to another reaches in the shortest time. (Brachistochrone problem).*

Solution. Let the particle slide on the curve OA from O with zero velocity. Let OP = s and time taken from 0 to P = t. By the law of conservation of energy, we have



K.E. at P – K.E. at O = potential energy at P.

$$\frac{1}{2}mv^2 - 0 = mgh$$

$$\Rightarrow \quad \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 = mgh \ \ or \ \ \frac{ds}{dt} = \sqrt{(2gy)}$$

Time taken by the particle to move from O to A

$$T = \int_0^T dt = \int_0^{x_1} \frac{ds}{\sqrt{(2gy)}} = \frac{1}{\sqrt{(2g)}}\int_0^{x_1}\frac{ds}{\sqrt{y}} = \frac{1}{\sqrt{(2g)}}\int_0^{x_1}\frac{\sqrt{(1+y'^2)}}{\sqrt{y}}dx$$

Here, $f = \dfrac{\sqrt{(1+y'^2)}}{\sqrt{y}}$ which is independent of x, i.e., $\dfrac{\partial f}{\partial x} = 0.$

and $\dfrac{\partial f}{\partial y'} = \dfrac{1}{2\sqrt{y}}\dfrac{2y'}{\sqrt{(1+y'^2)}} = \dfrac{y'}{\sqrt{y}\sqrt{(1+y'^2)}}$

Solution of Euler's equation is

$$f - y'\frac{\partial f}{\partial y'} = \text{constant } c$$

On substituting the values of f and $\dfrac{\partial f}{\partial y'}$, we get

$$\frac{\sqrt{(1+y'^2)}}{\sqrt{y}} - y'\frac{y'}{\sqrt{y}\sqrt{(1+y'^2)}} = c$$

$$\Rightarrow \quad \sqrt{1+y'^2} - \frac{y'^2}{\sqrt{(1+y^2)}} = c\sqrt{y} \ \text{ or } \ 1+y'^2 - y'^2 = c\sqrt{(1+y'^2)}\sqrt{y}$$

$$\Rightarrow \quad 1 = c\sqrt{y(1+y'^2)} \ \text{ or } \ 1+\left(\frac{dy}{dx}\right)^2 = \frac{1}{yc^2} \ \text{ or } \ \frac{dy}{dx} = \frac{\sqrt{1-yc^2}}{yc^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{1/c^2 - y}}{y} = \frac{\sqrt{a-y}}{y} \qquad \left(\frac{1}{c^2} = a\right)$$

$$dx = \sqrt{\frac{y}{a-y}}\,dy$$

$$\int_0^x x\,dx = \int_0^y \sqrt{\left(\frac{y}{a-y}\right)}\,dy \qquad \text{Put } y = a\sin^2\theta,\quad dy = 2a\sin\theta\cos\theta\,d\theta$$

$$x = \int_0^\theta \sqrt{\left(\frac{a\sin^2\theta}{a - a\sin^2\theta}\right)}\,2a\sin\theta\cos\theta\,d\theta = \int_0^\theta\left(\frac{\sin\theta}{\cos\theta}\right)2a\sin\theta\cos\theta d\theta = \int_0^\theta 2a\sin^2\theta d\theta$$

$$= a\int_0^\theta (1-\cos 2\theta)\,d\theta = a\left(\theta - \frac{\sin 2\theta}{2}\right)_0^\theta$$

$$\Rightarrow \quad x = \frac{a}{2}(2\theta - \sin 2\theta) \ \text{ and } \ y = a\sin^2\theta = \frac{a}{2}(1-\cos 2\theta)$$

On putting $\dfrac{a}{2} = A$ and $2\theta = \Theta$ $\left.\begin{aligned} x &= A(\Theta - \sin\Theta) \\ y &= A(1-\cos\Theta)\end{aligned}\right]$ which is a cycloid. **Ans.**

EXERCISE 38.1

1. Find the extremal of the functional

$$I[y(x)] = \int_{x_0}^{x_1} \frac{1+y^2}{y'^2}\,dy$$

2. Solve the Euler's equation for $\int_{x_0}^{x_1} (x+y')\,y'\,dx.$

3. Solve the Euler's equation for $\int_{x_0}^{x_1} (1+x^2y')\,y'\,dx$

Find the extremals of the functional and extremum value of the following:

4. $I[y(x)] = \int_{x_0}^{x_1} \frac{1+y^2}{y'^2}\,dx$

5. $I[y(x)] = \int_{\frac{1}{2}}^{1} x^2 y'^2\,dx$ subject to $y\left(\frac{1}{2}\right) = 1, y(1) = 2.$

6. $I[y(x)] = \int_0^2 (x-y')^2\,dx$ subject to $y(0) = 0, y(2) = 4.$

7. $\int_0^{\frac{\pi}{2}} (y'^2 - y^2)\,dx$ subject to $y(0) = 0,\ y\left(\frac{\pi}{2}\right) = 1$

8. $\int_0^1 (y'^2 + 12xy)\,dx$ subject to $y(0) = 0,\ y(1) = 1$

9. $\int_1^2 \frac{\sqrt{(1+y'^2)}}{x}\,dx$ subject to $y(1) = 0, y(2) = 1.$

ANSWERS

1. $y = \sinh(c_1 x + c_2)$	2. $y = -\frac{x^2}{4} + c_1 x + c_2$	3. $y = cx^{-1} + c^2$
4. $y = \sinh(c_1 x + c_2)$	5. $y = -\frac{c}{x} + d$, value = 1	6. $y = \frac{x^2}{2} + cx + d$, value = 2
7. $y = \sin x$, value = 0	8. $y = x^3$, value $= \frac{21}{5}$	9. $y = x^3$

38.6 ISOPERIMETRIC PROBLEMS

The determination of the shape of a closed curve of the given perimeter enclosing maximum area is the example of isoperimetric problem. In certain problems it is necessary to make a given integral.

$$I = \int_{x_1}^{x_2} f(x, y, y')\,dx \qquad ...(1)$$

maximum or minimum while keeping another integral

$$I = \int_{x_1}^{x_2} g(x, y, y')\,dx = K \text{ (Constant)} \qquad ...(2)$$

Problems of this type are solved by Lagrange's multipliers method. We multiply (2) by λ and add to (1) to extremize (1)

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$$I* = \int_{x_1}^{x_2} f(x, y, y')\,dx + \lambda \int_{x_1}^{x_2} g(x, y, y')\,dx = \int_{x_1}^{x_2} F\,dx \text{ (say)}$$

Then by Euler's equation $\quad \dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) l = 0$.

Note. Isoperimetric problem. To find out possible curves having the same perimeter, the one which encloses the maximum area.

Example 7. *Find the shape of the curve of the given perimeter enclosing maximum area.*

Solution. Let P be the perimeter of the closed curve,

Then
$$P = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\,dx \qquad ...(1)$$

The area enclosed by the curve, x-axis and two perpendicular lines is

$$A = \int_{x_1}^{x_2} y\,dx \qquad ...(2)$$

We have to find the maximum value of (2) under the condition (1).

By Lagrange's multiplier method.

$$f = y + \lambda\sqrt{1 + y'^2}$$

For maximum or minimum value of A, F must satisfy Euler's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$1 - \lambda\frac{d}{dx}\left[\frac{1}{2}(1 + y'^2)^{-\frac{1}{2}}(2y')\right] = 0 \text{ or } 1 - \lambda\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0$$

Integraing *w.r.t.*'x', we get $x - \dfrac{\lambda y'}{(1 + y'^2)} = a$

$$\Rightarrow \qquad \frac{\lambda y'}{\sqrt{(1 + y'^2)}} = x - a \text{ or } \lambda^2 y'^2 = (1 + y'^2)(x - a)^2$$

$$[\lambda^2 - (x - a)^2]\, y'^2 = (x - a)^2$$

$$\Rightarrow \qquad y' = \frac{x - a}{\sqrt{[\lambda^2 - (x - a)^2]}} \text{ or } \frac{dy}{dx} = \frac{x - a}{\sqrt{[\lambda^2 - (x - a)^2]}}$$

Integrating *w.r.t.* (x), we obtain

$$y = -\sqrt{[\lambda^2 - (x - a)^2]} + b$$

$$\Rightarrow y - b = -\sqrt{[\lambda^2 - (x - a)^2]} \Rightarrow (y - b)^2 = \lambda^2 - (x - a)^2 \Rightarrow (x - a)^2 + (y - b)^2 = \lambda^2$$

This is the equation of a circle whose centre is (a, b) and radius λ. **Ans.**

Example 8. *Find the extremal of the functional* $A = \int_{t_1}^{t_2} \frac{1}{2}(x\dot{y} - y\dot{x})\,dt$ *subject to the integral constraint* $\int_{t_1}^{t_2} \frac{1}{2}\sqrt{(\dot{x}^2 - \dot{y}^2)}\,dt = l.$

Solution. Here $f = \frac{1}{2}(x\dot{y} - \dot{x}y)$, $g = \sqrt{\dot{x}^2 - \dot{y}^2}$

$$F = f + \lambda g$$

$$F = \frac{1}{2}(x\dot{y} - y\dot{x}) + \lambda\sqrt{\dot{x}^2 + \dot{y}^2}$$

For A to have extremal F must satisfy the Euler's equation

$$\frac{\partial F}{\partial x} - \frac{d}{dx}\left[\frac{\partial F}{\partial \dot{x}}\right] = 0 \qquad ...(1)$$

$$\frac{\partial F}{\partial y} - \frac{d}{dt}\left[\frac{\partial F}{\partial \dot{y}}\right] = 0 \qquad ...(2)$$

From (1) $$\frac{1}{2}\dot{y} - \frac{d}{dt}\left(-\frac{y}{2} + \frac{\lambda 2\dot{x}}{2\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0$$

$$\frac{d}{dt}\left(y - \frac{\lambda\dot{x}}{2\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0 \qquad ...(3)$$

From (2) $$-\frac{1}{2}\dot{x} - \frac{d}{dt}\left[\frac{x}{2} + \frac{\lambda\dot{y}}{\sqrt{x^2 + y^2}}\right] = 0 \qquad ...(4)$$

$$\frac{d}{dt}\left[x - \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right] = 0$$

Integrating (3) and (4), we have

$$y - \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = c_1 \Rightarrow y - c_1 = \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \qquad ...(5)$$

$$x - \frac{\lambda y}{\sqrt{\dot{x}^2 + \dot{y}^2}} = c_2 \Rightarrow x - c_2 = \frac{\lambda y}{\sqrt{\dot{x}^2 + \dot{y}^2}} \qquad ...(6)$$

Squaring (5), (6) and adding, we get

$$(x - c_2)^2 + (y - c_1)^2 = \lambda^2\left(\frac{\dot{x}^2 + \dot{y}^2}{\dot{x}^2 + \dot{y}^2}\right)$$

$$(x - c_2)^2 + (y - c_1)^2 = \lambda^2$$

This is the equation of circle. **Ans.**

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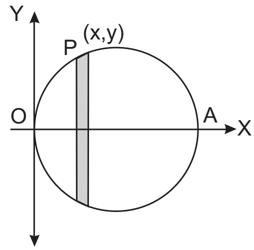
Example 9. *Find the solid of maximum volume formed by the revolution of a given surface area.*

Solution. Let the curve PA pass through origin and it is rotated about the x-axis.



$$S = \int_0^a 2\pi y\, ds$$

$$S = \int_0^a 2\pi y\sqrt{(1+y'^2)}\,dx \qquad ...(1)$$

$$V = \int_0^a \pi y^2\, dx \qquad ...(2)$$

Here we have to extremize V with the given S.

Here $$f = \pi y^2,\; g = 2\pi y\sqrt{(1+y'^2)}$$

$$F = f + \text{l}g$$

$$F = \pi y^2 + \lambda 2\pi y\sqrt{(1+y'^2)}$$

For maximum V, F must satisfy Euler's equation. But F does not contain x.

$\therefore$ $$F - y'\frac{\partial F}{\partial y'} = C$$

$\Rightarrow$ $$\pi y^2 + \lambda 2\pi y\sqrt{(1+y'^2)} - y'\frac{1}{2}\frac{2\pi y\,\lambda y'}{\sqrt{(1+y'^2)}} = C$$

$\Rightarrow$ $$\pi y^2 + 2\pi y\lambda\sqrt{(1+y'^2)} - \frac{2\pi\lambda y y'^2}{\sqrt{(1+y'^2)}} = C$$

$\Rightarrow$ $$\pi y^2 + \frac{2\pi y\lambda}{\sqrt{(1+y'^2)}} = C$$

As the curve passes through origin (0, 0), so $C = 0$.

$$\pi y^2 + \frac{2\pi y\lambda}{\sqrt{(1+y'^2)}} = 0$$

$\Rightarrow$ $$y + \frac{2\lambda}{\sqrt{(1+y'^2)}} = 0 \;\Rightarrow\; y\sqrt{(1+y'^2)} = -2\lambda$$

$\Rightarrow$ $$1 + y'^2 = \frac{4\lambda^2}{y^2} \qquad \Rightarrow\; y'^2 = \frac{4\lambda^2}{y^2} - 1 = \frac{4\lambda^2 - y^2}{y^2}$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{\sqrt{(4\lambda^2 - y^2)}}{y}$$

$$\int \frac{y\,dy}{\sqrt{(4\lambda^2 - y^2)}} = \int dx + C$$

$$-\sqrt{4\lambda^2 - y^2} = x + C \qquad ...(1)$$

$\Rightarrow$ $$\sqrt{4\lambda^2 - y^2} = -x - C$$

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The curve expasses through (0, 0). On putting $x = 0$, $y = 0$ in (1) we get

$$-C = 2\lambda$$

(1) becomes $\sqrt{4\lambda^2 - y^2} = -x + 2\lambda$

Squaring $4\lambda^2 - y^2 = (x - 2\lambda)^2$

$\Rightarrow$ $(x - 2\lambda)^2 + y^2 = 4\lambda^2$

This is the equation of a circle.

Hence, on revolving the circle about x-axis, the solid formed is a sphere. **Ans.**

EXERCISE 38.2

1. Show that an isosceles triangle has the smallest perimeter for a given area and a given base.
2. Find the extremal in the isoperimetric problem of the extremum of

$$\int_0^1 (y'^2 + z'^2 - 4xz' - 4z)\, dx$$

subject to $\int_0^1 (y'^2 + xy' - z'^2)\, dx = 2$, $y(0) = 0$, $z(0) = 0$, $y(1) = 1$, $z(1) = 1$.

3. Find the surface with the smallest area which encloses a given volume.
4. Find the extremal of the functional $\int_{t_1}^{t_2} \sqrt{x^2 + y^2 + z^2}\, dt$ subject to $x^2 + y^2 + z^2 = a^2$
5. Find the extremals of the isoperimetric problem $\int_{x_0}^{x_1} y'^2\, dx$ subject to $\int_{x_0}^{x_1} y\, dx = c$.

ANSWERS

2. $y = \frac{-5x^2}{2} + \frac{7x}{2}, z = x.$
3. Sphere
4. Arc of a great circle of a sphere.
5. $y = x^2 + ax + b$

38.7 FUNCTIONALS OF SECOND ORDER DERIVATIVES

Let us consider the extremum of a functional.

$$\int_{x_1}^{x_2} [f(x, y, y', y'')]\; dx \qquad ...(1)$$

The necessary condition for the above mentioned functional to be extremum is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0$$

Proof. Let the boundary conditions be

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y'(x_1) = y'_1,\ y'(x_2) = y'_2$$

Let α be a parameter and $\eta(x)$ is a differentiable function.

At the end points $\eta(x_1) = \eta(x_2) = 0$ and $\eta'(x_1) = \eta'(x_2) = 0$

Putting $y + \alpha\eta(x)$ for y in (1), we have

$$\int_{x_1}^{x_2} f[x, y+\alpha\,\eta(x), y'+\alpha\,\eta'(x), y''+\alpha\,\eta''(x)]\,dx$$

Writing $\int_{x_1}^{x_2} f[x, y+\alpha\,\eta(x), y'+\alpha\,\eta'(x), y''+\alpha\,\eta''(x)]\,dx = \int_{x_1}^{x_2} F dx = 1$

For extremum value of (1)

$$\frac{dI}{d\alpha} = 0 \qquad \frac{dI}{d\alpha} = \int_{x_1}^{x_2} \frac{\partial F}{\partial \alpha} dx$$

Differentiating under the sign of integral, we get

$$= \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial \alpha} + \frac{\partial F}{\partial y''}\frac{\partial y''}{\partial \alpha}\right)dx = \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\frac{\partial(\alpha\,n)}{\partial \alpha} + \frac{\partial F}{\partial y'}\frac{\partial(\alpha\,n')}{\partial \alpha} + \frac{\partial F}{\partial y''}\frac{\partial(\alpha\,n'')}{\partial \alpha}\right)dx$$

But $\frac{dI}{d\alpha} = 0$ when $\alpha = 0$

$$0 = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y}\eta' + \frac{\partial f}{\partial y}\eta''\right]dx \text{ or } \int_{x_1}^{x_1}\frac{\partial f}{\partial y}\eta\,dx + \int_{x_1}^{x_2}\frac{\partial f}{\partial y'}\eta'\,dx + \int_{x_1}^{x_2}\frac{\partial f}{\partial y''}\eta''\,dx = 0$$

Integrating by parts, *w. r. t.* 'x', we have

$$\int_{x_1}^{x_2}\frac{\partial f}{\partial y}\eta\,dx + \left[\frac{\partial f}{\partial y}\eta - \int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right).\eta\,dx\right]_{x_1}^{x_2} + \left(\frac{\partial f}{\partial y''}\eta' - \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right).\eta + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)\int_{x_1}^{x_2}\eta\,dx\right)_{x_1}^{x_2} = 0$$

But $n(x_1) = n(x_2) = 0$ and $\eta'(x_1) = \eta'(x_2) = 0$

so $$\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)\right]\eta(x)\,dx = 0 \Rightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0$$ **Proved.**

EXERCISE 38.3

1. Find the extremal of $\int_{x_0}^{x_1}(16y^2 - y''^2 + x^2)\,dx$.

2. Find the extremal of $\int_{-c}^{c}\left(ay + \frac{1}{2}by''^2\right)dx$ subject to $y(-c) = 0$, $y'(-c) = 0$, $y(c) = 0$, $y'(c) = 0$.

3. Find the extremal of $\int_0^{\pi} y''^2\,dx$ subject to $\int_0^{\pi} y^2\,dx = 1$, $y(0) = y(\pi) = 0$, $y''(0) = y''(\pi) = 0$.

4. Find the extremal of $\int_{x_0}^{x_1}(2xy + y'''^2)\,dx$.

ANSWERS

1. $y = c_l e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$ **2.** $y = -\dfrac{a}{24b}(x^2 - c^2)^2$

3. $y = a_1 \sin x + a_2 \sin 2x + ...$ **4.** $y = \dfrac{x^7}{7!} + c_1 x^5 + c_2 x^4 + c_3 x^3 + c_4 x^2 + c_5 x + c_6$

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